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GEOMETRIC THEOREMS IN DYNAMICS

Edward Kasner and John De Cicco

1. *The equations of motion.* Consider the motion of a particle in the plane under the action of a positional field of force. If (ϕ,ψ) are the rectangular components of the force vector acting at any point (x,y), the equations of motion are

$$(1) \qquad \ddot{x} = \phi(x,y), \quad \ddot{y} = \psi(x,y).$$

Since the mass of the particle is constant, there is no loss of generality in assuming the mass to be unity. The components ϕ and ψ are assumed to be continuous and to possess continuous partial derivatives of the first and second orders with respect to x and y in a certain region of the (x,y)-plane. We shall use dots to indicate total differentiation with respect to the time t, primes to indicate total differentiation with respect to x, and subscripts x and y to indicate partial differentiation. We shall omit the trivial case where the force vector is identically zero in which case the trajectories consist merely of straight lines.

The particle may be started from any position (x_0,y_0) with any velocity $(\dot{x}_0,\dot{y}_0)$. A definite trajectory is then described. Since the same curve may be obtained by starting from any one of its ∞^1 points, the total number of trajectories for all initial conditions is ∞^3.

The differential equation of the third order representing this system of ∞^3 trajectories, found by eliminating the time t from (1), is

$$(2) \qquad (\psi - y'\phi)y''' = [\psi_x + (\psi_y - \phi_x)y' - \phi_y y'^2]y'' - 3\phi y''^2.$$

Kasner has given a characteristic set of *five geometric properties* of the ∞^3 dynamical trajectories of a positional field of force in his Princeton colloquium Lectures. It is our purpose to give new proofs of some of these theorems and also to give some new supplemdntary theorems.

2. *Osculating circle and osculating parabola.* Before stating and proving the theorems, it is necessary to collect the various formulas needed for our work.

The center (X,Y) and radius R of the osculating circle at any point of the trajectory $x = x(t)$, $y = y(t)$, are

$$(3) \qquad X + iY = x + iy + \frac{i(\dot{x} + i\dot{y})(\dot{x}^2 + \dot{y}^2)}{\dot{x}\ddot{y} - \ddot{x}\dot{y}}, \qquad R = \frac{(\dot{x}^2 + \dot{y}^2)^{3/2}}{\dot{x}\ddot{y} - \ddot{x}\dot{y}}.$$

In running coordinates (X,Y), the equation of the osculating

parabola is

$$[(\dot{x}\dddot{y}-\dddot{x}\dot{y})\{\dot{y}(X-x)-\dot{x}(Y-y)\}-3(\dot{x}\ddot{y}-\ddot{x}\dot{y})\{\ddot{y}(X-x)-\ddot{x}(Y-y)\}]^2+ \tag{4}$$
$$18(\dot{x}\ddot{y}-\ddot{x}\dot{y})^3\,[\dot{y}(X-x)-\dot{x}(Y-y)]=0.$$

The focus (X,Y) is

$$X+iY=x+iy+\frac{3(\dot{x}+i\dot{y})^2\,(\dot{x}\ddot{y}-\ddot{x}\dot{y})}{2[(\dot{x}+i\dot{y})(\dot{x}\dddot{y}-\dddot{x}\dot{y})-3(\ddot{x}+i\ddot{y})(\dot{x}\ddot{y}-\ddot{x}\dot{y})]}. \tag{5}$$

The equation of the directrix in running coordinates (X,Y) is

$$2(\dot{x}\dddot{y}-\dddot{x}\dot{y})[\dot{x}(X-x)+\dot{y}(Y-y)]-6(\dot{x}\ddot{y}-\ddot{x}\dot{y})[\ddot{x}(X-x)+\ddot{y}(Y-y)]= \tag{6}$$
$$3(\dot{x}^2+\dot{y}^2)(\dot{x}\ddot{y}-\ddot{x}\dot{y}).$$

3. *The osculating circles and the osculating parabolas of a dynamical family of ∞^3 trajectories.* Let v denote the magnitude and θ the inclination of the velocity vector so that $\dot{x}=v\cos\theta$, $\dot{y}=v\sin\theta$. The speed is $v=(\dot{x}^2+\dot{y}^2)^{1/2}$. Now we proceed to discuss the osculating circles and the osculating parabolas of the dynamical family (1).

The center (X,Y) and radius R of the osculating circles are

$$X+iY=x+iy+\frac{i(\cos\theta+i\sin\theta)v^2}{\psi\cos\theta-\phi\sin\theta},\qquad R=\frac{v^2}{\psi\cos\theta-\phi\sin\theta}. \tag{7}$$

In running coordinates (X,Y), the equation of the osculating circles is

$$(\psi\cos\theta-\theta\sin\theta)[(X-x)^2+(Y-y)^2]+ \tag{8}$$
$$2v^2[(X-x)\sin\theta-(Y-y)\cos\theta]=0.$$

The equation of the osculating parabolas is

$$\begin{bmatrix} v^2\{\psi_x\cos^2\theta+(\psi_y-\phi_x)\cos\theta\,\sin\theta-\phi_y\sin^2\theta\} \\ \{(X-x)\sin\theta-(Y-y)\cos\theta\}+3(\psi\cos\theta-\phi\sin\theta) \\ \{-\psi\,(X-x)+\phi\,(Y-y)\} \end{bmatrix}^2 + \tag{9}$$
$$18v^2(\psi\cos\theta-\phi\sin\theta)^3\,[(X-x)\sin\theta-(Y-y)\cos\theta]=0.$$

The focus (X,Y) is

$$X+iY=x+iy+\frac{3v^2(\cos\theta+i\sin\theta)^2\,(\psi\cos\theta-\phi\sin\theta)}{2\begin{bmatrix} v^2(\cos\theta+i\sin\theta)\{\psi_x\cos^2\theta+(\psi_y-\phi_x)\cos\theta\sin\theta- \\ \phi_y\sin^2\theta\}-3(\phi+i\psi)(\psi\cos\theta-\phi\sin\theta)\end{bmatrix}} \tag{10}$$

The equation of the directrices is

(11) $$2v^2[\psi_x \cos^2\theta + (\psi_y - \phi_x)\cos\theta \sin\theta - \phi_y \sin^2\theta][(X-x)\cos\theta + (Y-y)\sin\theta] - 6(\psi\cos\theta - \phi\sin\theta)[\phi(X-x) + \psi(Y-y)] = 3v^2(\psi\cos\theta - \phi\sin\theta).$$

4. *Actual and virtual trajectories.* If we consider the motion of a cannon ball in a given vertical plane under the action of gravity assumed constant, the triply infinite system of trajectories consists of parabolas with vertical axes. We do not, however, obtain all vertical parabolas, represented by the differential equation of the system of trajectories, which is here $y''' = 0$, but only those whose concavity is directed downwards. The other vertical parabolas, with concavity directed upwards, satisfy the same differential equation, and it is therefore convenient to include them in the system studied. We thus have a distinction of *actual* and *virtual* trajectories. The latter are the actual trajectories corresponding to gravity reversed in direction.

In an arbitrary field of force the same distinction arises. The complete system of trajectories is composed of the actual trajectories corresponding to the given force, and the virtual trajectories which are the actual trajectories corresponding to the reversed field. It is obvious that the system of trajectories is not changed if the force acting at every point is multiplied by a constant. If we were considering only actual trajectories, it would be necessary to restrict this constant to positive values, but as we include both actual and virtual, the constant factor may also be negative. (Of course, the constant must not be zero, since then the force would vanish and we should obtain the trivial system of straight lines).

It is easy to show that the virtual trajectories corresponding to the given field may be found by giving the initial speed of the particle a pure imaginary value. The cannon ball could be made to describe a parabola with its concavity directed upwards if only some kind of powder could be invented which would cause its initial speed to be imaginary!

In what follows, we shall discuss both the complete systems and the actual systems of trajectories.

5. *The Geometric Property* I. Let us now keep the lineal-element $E(x,y,\theta)$ fixed and vary the speed v. Corresponding to each value of the initial speed v, there is a definite trajectory described. Thus in all there are ∞^1 trajectories which pass through the lineal-element E. Upon constructing the osculating parabola to each one of these trajectories, we obtain a set of ∞^1 osculating parabolas. These are defined by (9) where v is an arbitrary parameter. The corresponding foci and directrices are given by (10) and (11). The Geometric Property I can be given in three equivalent forms.

(Ia) *The locus of the foci of the* ∞^1 *osculating parabolas which pass through the lineal-element E for the complete system of trajectories is*

$$(12)\qquad 2[\psi_x \cos^2\theta + (\psi_y - \phi_x)\cos\theta\,\sin\theta - \phi_y \sin^2\theta][(X-x)^2 + (Y-y)^2] + 3[(\phi\sin 2\theta - \psi\cos 2\theta)(X-x) - (\phi\cos 2\theta + \psi\sin 2\theta)(Y-y)] = 0.$$

This is a circle which passes through the point of E. Hereafter this locus shall be referred to as the focal circle. The center (X,Y) *and radius R of this focal circle are*

$$(13)\qquad \begin{aligned} X + iY &= x + iy + \frac{3i(\cos 2\theta + i\sin 2\theta)(\phi - i\psi)}{4[\psi_x\cos^2\theta + (\psi_y - \phi_x)\cos\theta\,\sin\theta - \phi_y\sin^2\theta]}, \\ R &= \frac{3(\phi^2 + \psi^2)^{1/2}}{4[\psi_x\cos^2\theta + (\psi_y - \phi_x)\cos\theta\,\sin\theta - \phi_y\sin^2\theta]}. \end{aligned}$$

From (12) and (13) it is seen that the focal circle can never reduce to a point. However it may degenerate into a straight line.

The focal circle degenerates into a straight line for every lineal-element E if and only if the components are

$$(14)\qquad \phi = ax + h, \qquad \psi = ay + k,$$

where (a,h,k) *are constants. That is, the field of force is either constant (Galileo) if* $a = 0$, *or elastic if* $a \neq 0$.

Omitting the special constant and elastic fields of force, it is found that *the focal circles degenerate into straight lines along the lineal-elements of the net of* $2\,\infty^1$ *curves*

$$(15)\qquad \psi_x + (\psi_y - \phi_x)y' - \phi_y y'^2 = 0.$$

This net may be real or imaginary or consist of only ∞^1 *curves.*

It is remarked that *a field of force is conservative if and only if the net* (15) *is an orthogonal net.*

For the actual trajectories, it is seen by (10) that the foci of the osculating parabolas do not describe the whole focal circle. As v varies from zero to infinity, *the foci of the osculating parabolas constructed at the lineal-element E, describe an arc of the focal circle beginning with the point* $P(x,y)$ *of E and terminating with the point* $Q(X,Y)$ *defined by*

$$(16)\qquad Q:\ X + iY = x + iy + \frac{3(\cos\theta + i\sin\theta)(\psi\cos\theta - \phi\sin\theta)}{2[\psi_x\cos^2\theta + (\psi_y - \phi_x)\cos\theta\,\sin\theta - \phi_y\sin^2\theta]}.$$

It is noted that the points P and Q are the points of intersection of

the focal circle with the line determined by E. The foci of the osculating parabolas of the virtual trajectories describe the remaining arc of the focal circle.

(Ib) *For a given lineal-element E, the directrices of the ∞^1 osculating parabolas pass through a common point $D(X,Y)$ defined by*

$$(17)\qquad D: X+iY = x+iy - \frac{3i(\phi+i\psi)}{2[\psi_x\cos^2\theta+(\psi_y-\phi_x)\cos\theta\,\sin\theta-\phi_y\sin^2\theta]}.$$

Only for the constant or elastic fields of force do the directrices form a parallel pencil for every lineal-element E. Omitting these special cases, it is seen that the directrices form a parallel pencil corresponding to the lineal-elements of the net (15).

For the actual trajectories, the directrices of the osculating parabolas describe the limited angle of lines whose initial line is the line DP : $\phi(X-x)+\psi(Y-y)=0$, and whose terminal line is the line DQ defined by the equation

$$(18)\qquad 2[\psi_x\cos^2\theta+(\psi_y-\phi_x)\cos\theta\,\sin\theta-\phi_y\sin^2\theta][(X-x)\cos\theta+(Y-y)\sin\theta] = 3(\psi\cos\theta-\phi\sin\theta).$$

The line DP is perpendicular to the force vector at P and the line DQ is perpendicular to the line PQ. The acute angle formed by the lines DP and DQ is numerically equal to the acute angle formed by the given lineal element and the force vector.

(Ic). *The ∞^1 osculating parabolas which pass through the lineal-element E for the complete system of trajectories are tangent to the straight line*

$$(19)\qquad 2[\psi_x\cos^2\theta+(\psi_y-\phi_x)\cos\theta\,\sin\theta-\phi_y\sin^2\theta][-\psi(X-x)+\phi(Y-y)]+3(\psi\cos\theta-\phi\sin\theta)^2=0.$$

This straight line is tangent to the focal circle at Q and is parallel to the force vector at P.

For the constant or elastic fields of force, this line is at infinity. Except for these special cases, it is seen that this is the line at infinity only for the lineal-elements of the net (15).

For the actual trajectories, the ∞^1 osculating parabolas are tangent to the half-line (19) with initial point Q.

6. *Property* II. This may be stated in any one of the following three equivalent ways.

(IIa) *The circle that corresponds, according to Property* I, *to a*

lineal-element E, is so situated that the element bisects the angle between the tangent to the focal circle and the direction of the force vector.

This result is also valid for the constant and elastic fields and also along the net (15) of other fields.

(IIb) *As the lineal-element E rotates about the fixed point P, the point D given by* (17) *describes the line DP which is perpendicular to the force vector at P.*

(IIc) *As the lineal-element E rotates about the fixed point P, the lines* (19) *describe a parallel pencil of lines all parallel to the force vector at P.*

The Properties (IIb) and (IIc) are valid for all fields of force except the constant and elastic fields provided the lineal-elements of the net (15) are avoided.

7. *Property* III. We proceed to discuss the three equivalent forms of this Property III. It is found that the first form is of more geometric interest than the other two forms.

At any lineal-element E, we proceed to determine the speed v of those trajectories that have four-point contact with its circle of curvature. By differentiating the radius of curvature R as given by (7), setting the result equal to zero, and using the conditions

$$\frac{v^2}{R} = N = \psi\cos\theta - \phi\sin\theta, \quad v\frac{dv}{ds} = T = \phi\cos\theta + \psi\sin\theta, \tag{20}$$

where N and T are the normal and tangential components of the force vector, it is found that the appropriate speed v is

$$v^2 = \frac{3(\psi\cos\theta - \phi\sin\theta)(\phi\cos\theta + \psi\sin\theta)}{\psi_x\cos^2\theta + (\psi_y - \phi_x)\cos\theta\,\sin\theta - \phi_y\sin^2\theta}. \tag{21}$$

Thus through any lineal-element E, there is only one trajectory which is hyperosculated by its circle of curvature. This trajectory may be actual or virtual. Substituting (21) into (7), it is found that this circle has as center and radius

$$\begin{gathered} X + iY = x + iy + \frac{3i(\cos\theta + i\sin\theta)(\phi\cos\theta + \psi\sin\theta)}{\psi_x\cos^2\theta + (\psi_y - \phi_x)\cos\theta\,\sin\theta - \phi_y\sin^2\theta}, \\ R = \frac{3(\phi\cos\theta + \psi\sin\theta)}{\psi_x\cos^2\theta + (\psi_y - \phi_x)\cos\theta\,\sin\theta - \phi_y\sin^2\theta}. \end{gathered} \tag{22}$$

The equation of this circle of curvature is

$$(23)\qquad [\psi_x \cos^2\theta + (\psi_y - \phi_x)\cos\theta\ \sin\theta - \phi_y \sin^2\theta][(X-x)^2 + (Y-y)^2] + 6(\phi\cos\theta + \psi\sin\theta)[(X-x)\sin\theta - (Y-y)\cos\theta] = 0\,.$$

For the constant and elastic fields of force, or along the lineal-elements E of the net (15), this speed v of (21) may be infinite or indeterminate, and the circle (23) reduces into the straight line of the lineal-element E.

As an example, consider the Newtonian field of force where the center of attraction is at the origin and the magnitude of the force vector varies inversely as the square of the distance $r = (x^2 + y^2)^{1/2}$ from the origin. In canonical form, the rectangular components of the force vector are $\phi = -x/r^3$ and $\psi = -y/r^3$. Of course, the trajectories are conic sections with one focus at the origin. The net (15) consists of the lines of force (the lines through the origin) and their orthogonal trajectories (the circles with centers at the origin). At any lineal-element E, not on the net (15), the speed v of (21) is found to be $v^2 = 1/r$. The trajectory which is hyperosculated by its circle of curvature is the ellipse whose center is the point of intersection of the normal to the lineal-element E and the line through the origin parallel to the direction of E. That is, E is the lineal-element at an end point of the minor axis of the ellipse. On the other hand, if the lineal-element E belongs to the net (15), the speed v of (21) is indeterminate. If E is tangent to a circle with center at the origin, all the trajectories through E are conic sections, all of which have a vertex at E. Thus these conics are all hyperosculated by their circles of curvature at E. Finally if E is on a line through the origin, all the trajectories consist of this single line. In conclusion, it is noticed that the initial speed for a parabolic trajectory is $v^2 = 2/r$. Thus the ratio of the speed of the hyperosculated trajectory to that of the parabolic trajectory is $1/\sqrt{2}$.

(IIIa) *In each direction at a given point, there is one trajectory which has four-point contact with its circle of curvature provided the two directions of the net* (15) *are excluded. The locus of the centers of the* ∞^1 *hyperosculating circles constructed at the given point* P *is the conic section*

$$(24)\qquad \phi_y(X-x)^2 + (\psi_y - \phi_x)(X-x)(Y-y) - \psi_x(Y-y)^2 + 3[-\psi(X-x) + \phi(Y-y)] = 0\,,$$

passing through that point in the direction of the force vector.

In general, the envelope of the hyperosculating circles at the fixed point P is a bicircular quartic with a cusp at P in the direction orthogonal to that of the force vector at P.

The field of force is conservative if and only if the conic section (24) *is a rectangular hyperbola.*

The conic (24) is degenerate if and only if

$$\psi_x\psi^2 - (\psi_y - \phi_x)\phi\psi - \phi_y\phi^2 = 0 . \tag{25}$$

In that event, the conic (24) is composed of the straight line determined by the force vector and the straight line

$$\phi_y\phi(X-x) + \psi_x\psi(Y-y) - 3\phi\psi = 0 . \tag{26}$$

The conic section (24) *constructed at any point P is degenerate if and only if the direction orthogonal to that of the force vector at P is on the net* (15).

The straight line (26) is perpendicular to the straight line determined by the force vector if and only if the field of force is conservative.

It is found that *the envelope of the focal circles* (12) *constructed at a fixed point P is the circle*

$$\begin{gathered}[(\psi_y - \phi_x)^2 + 4\phi_y\psi_x][(X-x)^2 + (Y-y)^2] - \\ 6[\phi(\phi_x - \psi_y) + \psi(\phi_y + \psi_x)](X-x) - 6[-\psi(\phi_x - \psi_y) + \\ \phi(\phi_y + \psi_x)](Y-y) + 9(\phi^2 + \psi^2) = 0 ,\end{gathered} \tag{27}$$

in general position.

This circle degenerates into a straight line if and only if the two directions defined by the net (15) at the point P coincide.

The center (X,Y) and radius R of the circle (27) are

$$\begin{gathered}X + iY = x + iy + \frac{3(\phi - i\psi)(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})(\phi + i\psi)}{(\psi_y - \phi_x)^2 + 4\phi_y\psi_x} , \\ R = \frac{3(\phi_y - \psi_x)(\phi^2 + \psi^2)^{1/2}}{(\psi_y - \phi_x)^2 + 4\phi_y\psi_x} .\end{gathered} \tag{28}$$

The circle (27) *reduces to a point if and only if the field of force is conservative. Thus only in the conservative case do all the focal circles at any fixed point form a pencil.*

The center of the circle (27) is at the given point P if and only if the field of force is of the Lecornu type: $\phi_x = \psi_y$, $\phi_y = -\psi_x$.

By the equations (13), it can be established that *the locus of the centers of the focal circles constructed at the point P is the*

conic section

$$(29)\quad \begin{aligned}&(\phi_y-\psi_x)(\phi^2+\psi^2)^{1/2}\,[(X-x)^2+(Y-y)^2]^{1/2} = \\ &[\phi(\phi_x-\psi_y)+\psi(\phi_y+\psi_x)](X-x)+[-\psi(\phi_x-\psi_y)+\phi(\phi_y+\psi_x)](Y-y) - \\ &\frac{3}{2}(\phi^2+\psi^2),\end{aligned}$$

with one focus at the fixed point P.

The circle (27) is the *director circle* of this conic (29). That is, the circle is the locus of points from which mutually orthogonal tangent lines to the conic can be drawn.

This conic (29) reduces to a straight line if and only if the field of force is conservative. It is a circle with center at the given point P if and only if the field of force is of the Lecornu type.

As the lineal-element E is rotated about the fixed point P, the point Q of (16) *describes the conic section*

$$(30)\quad \begin{aligned}&2[\psi_x(X-x)^2+(\psi_y-\phi_x)(X-x)(Y-y)-\phi_y(Y-y)^2] + \\ &3[-\psi(X-x)+\phi(Y-y)]=0,\end{aligned}$$

which passes through the point P in the direction of the force vector at P. This conic never coincides with the conic (24).

The conic (30) is degenerate if and only if

$$(31)\quad \psi_x\phi^2+(\psi_y-\phi_x)\phi\psi-\phi_y\psi^2=0.$$

Then the conic (30) consists of the straight line determined by the force vector and the straight line

$$(32)\quad 2\psi_x\phi(X-x)+2\phi_y\psi(Y-y)-3\phi\psi=0.$$

The conic section (30) *constructed at any point P is degenerate if and only if the direction of the force vector at P is on the net* (15).

The straight line (32) is perpendicular to the straight line determined by the force vector if and only if the field of force is conservative.

Evidently *the conic sections* (30) *are rectangular hyperbolas if and only if the field of force is conservative.*

The following is an equivalent form of Property III.

(IIIb) *By Property* IIb, *the point D given by* (17) *describes the line DP as the lineal-element E rotates about the fixed point P. The correspondence between the range of the points D and the pencil of*

lineal-elements E through P is one-to-two of the special form

$$d = \frac{3(\phi^2 + \psi^2)^{\frac{1}{2}}}{2[\psi_x \cos^2\theta + (\psi_y - \phi_x)\cos\theta \ \sin\theta - \phi_y \sin^2\theta]}, \tag{33}$$

where d denotes the distance DP and θ is the inclination of the lineal-element E.

A final equivalent form is the following one.

(IIIc) *By Property* IIc, *the lines* (19) *form a parallel pencil of straight lines, all of which are parallel to the force vector at P. The correspondence between the parallel pencil of lines* (19) *and the pencil of lineal-elements E through P is one-to-two of the special form*

$$d = \frac{3(\psi\cos\theta - \phi\sin\theta)^2}{2(\phi^2+\psi^2)^{\frac{1}{2}}\ [\psi_x \cos^2\theta + (\psi_y - \phi_x)\cos\theta \ \sin\theta - \phi_y \sin^2\theta]}, \tag{34}$$

where d is the distance between any line (19) *and the line of the force vector at P, and θ is the inclination of the lineal-element E.*

It is noted that the lines DQ as defined by the equation (18) envelope a unicursal curve of fourth degree as the lineal-element E rotates about the fixed point P.

8. *Property* IV. The normal to the conic section (24) at the point P intersects the conic again in the point N given by

$$N : X + iY = x + iy + \frac{3i(\phi - i\psi)(\phi^2+\psi^2)}{\psi_x\phi^2 + (\psi_y - \phi_x)\phi\psi - \phi_y\psi^2} \tag{35}$$

The distance between P and N is equal to three times the radius of curvature of the line of force at P.

Property IV. *With any point P, there is associated a certain conic section* (24) *passing through it as described in property* IIIa. *The normal to the conic section at P cuts the conic again at a distance equal to three times the radius of curvature of the line of force passing through P.*

If the direction of the line of force at P is on the net (15), the point N is at infinity. The radius of curvature of the line of force at that point P is infinite. Therefore the Property IV as stated above is still valid in this case in a trivial sense. Of course, if the direction of the line of force is on the net (15) for all points P, then the lines of force are straight lines.

9. *Property* V. Consider the conic (24) of Property IIIa which passes through the point P. Take any two fixed perpendicular directions for the x direction and the y direction. Through P, draw lines in these directions meeting the conic again at A and B respectively. Also construct the normal at P meeting the conic again at N. At A draw a line in the y direction meeting this normal in some point A', and at B draw a line in the x direction meeting the normal in some point B'.

Property V. *When the point P is moved, the distances AA' and BB' and the slope ω of the lines of force relative to the chosen x direction change in the following manner*

$$\frac{\partial}{\partial x}\frac{1}{AA'} + \frac{\partial}{\partial y}\frac{1}{BB'} + \frac{1}{3\omega^2}(\omega\omega_{xy} - \omega_x\omega_y) = 0.$$

This is valid for any pair of orthogonal directions.

Kasner has proved that these five properties, as outlined above, are characteristic. That is, these are sufficient as well as necessary for a system of ∞^3 curves to represent the dynamical trajectories of a field of force. The field of force is determined up to an arbitrary constant factor.

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ANNOUNCEMENT

The Mathematics Magazine takes pleasure in making the following announcement received from Prof. Dugué of Caen, France concerning a scientific jubilee honoring our colleague *Prof. Maurice Frechét*, distinguished mathematician of the Institut Henri Poincaré, Paris.

Professor Maurice Frechét approaches the age for retirement. His friends, colleagues and students wish at this occasion to show their admiration, their attachment and their gratitude by celebrating his scientific jubilee.

Following the wishes of Maurice Fréchet, this celebration will be characterized by great simplicity. At the beginning of the year 1950, one or two prizes will be awarded to the author or authors of a work in general analysis. The winners will be selected by a committee nominated by the Council of the Mathematical Society of France and their names will be proclaimed at an ordinary meeting of the Society. Following this meeting the bulletin of the Society will publish the names of the winners and the list of subscribers.

To retain the note of simplicity, this letter has been signed only by the members of the geometry section of the Academy of Science, the members of the Council of the Mathematical Society and the present or former mathematical colleagues of Prof. Fréchet at the Faculty of Science at College of France and at the Polytechnic School.

E. BOREL, E. CARTAN, DENJOY, HADAMARD, JULIS, MONTEL.

BRARD, PRESIDENT; BAYARD, BELGODÈRE, BENOIT, BOOS, CAGNAC, H. CARTAN, CHATELET, CHOQUET, COURBON, COURTAND, DESFORGE, MME DUBREIL, FOURÈS, JANET, JEAN, LAMOTHE, LELONG, LERAY, LICHNEROWICZ, MAILLARD, MANDELBROJT, MARCHAND, SCHWARTZ.

BÉGHIN, BOREL, BOULIGAND, BRARD, DE BROGLIE, E. CARTAN, H. CARTAN, CHAPELON, CHATELET, CHAZY, G. DARMOIS, DENJOY, DUBREIL, FAVARD, GARNIER, HADAMARD, JANET, JULIA, LERAY, P. LÉVY, MANDELBROJT, MONTEL, PÉRÈS, PLATRIER, ROY, THIRY, VALIRON, VESSIOT, VILLAT.

To these names will be added, after the publication of the bulletin, those of foreign friends of Maurice Frechét, who will have given their help in the collection of subscriptions.

All subscriptions should be addressed to: M. Daniel Dugué, Professeur à la Faculté des Sciences, 52, rue d'Authie, à Caen, Calvados, France (Postal check account: Rouen, 131-147).

Rules for the contest:—The manuscripts in a foreign language should be accompanied by a typewritten outline in French. All the manuscripts should have reached the Président de la Société mathématique, Institut H. Poincaré, 11 rue Pierre Curie, Paris (5^e), before the first of March, 1950; they should all be works in general analysis (theory of abstract spaces, transformation of abstract elements into abstract elements) or its applications.

The manuscripts should carry the name and address of the author. However, authors who wish to remain anonymous may write at the end of the manuscript a sentence that has been reproduced on a sealed envelope containing their name and their address. The envelopes corresponding to the memoirs not chosen will be destroyed without being opened.

A NEW GENERAL METHOD OF SUMMING DIVERGENT SERIES

Glenn James

Foreword. The practice, in more advanced mathematics, of extending the meaning of terms already in use in elementary mathematics has so modified the meaning of the word *sum* that its friends in algebra could not be expected to recognize it without a reintroduction. Out of the multitude of usages of this word we are concerned here only with its use as the *sum of an infinite series*, meaning by infinite series an indicated sum of an unlimited number of terms, such as .9 .09 .009 .0009 · · · .

In the study of infinite series (or just *series* as it is customary to call them), we mean by *sum* not the result of adding the terms as in algebra (an impossible feat since there are an unlimited number of them) but the limit of the algebraic sum of the first n terms as n increases beyond all bounds if this limit exists, in other words the limit of the sequence of its partial sums. (And this limit can often be found from the nature of the sequence). In the above example, the partial sums are .9 .99 .999 · · · and the limit of these partial sums is, of course, 1. As a further example consider the series $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$. This series has for its sequence of partial sums $\frac{1}{2},\frac{3}{4},\frac{7}{8},\cdots,\frac{2^n-1}{2^n},\cdots$. Hence its sum is the limit of $\frac{(2^n-1)}{2^n}$. Writing this quantity in the form $(1-\frac{1}{2^n})$ it becomes evident that it approaches 1 as n increases. This limit is the best value one can choose for the sum of the series. For if one goes half way across a room then half of the remaining distance, then half of the remaining distance, and so on indefinitely, the width of the room is certainly the best value to take for the sum of the intervals traveled. For if one would choose any point short of the other side of the room the traveler would soon be nearer to the opposite wall than to this point.

But the sequence of partial sums does not always have a limit. For example the series $1+1+1+\cdots$ has for its partial sums $1,2,3,4,\cdots,n,\cdots$ which increase beyond all bounds; and the series $1-1+1-1+\cdots$ has for its partial sums $1,0,1,0,1,\cdots$, which *oscillate* between 1 and 0.

When the sequence of partial sums of a series has a limit the series is said to be *convergent*, when this sequence does not have a limit the series is said to be *divergent*. The classic methods of summing divergent series have been designed for summing divergent series whose partial sums oscillate.

It seems desirable that any definition of the sum of a divergent series when applied to a convergent series should give the previously defined sum of that series, namely the limit of its sequence of partial sums. When a definition has this property it is said to be *regular*.

So we study the sums of convergent series and formulate definitions of the sum of divergent series upon the basis of our findings[1]. This procedure can be made clear by the use of a graph. Suppose we plot the partial sums $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \cdots$ as ordinates corresponding to the respective values of n as abscissas, then draw a smooth curve through these points (Fig. 1).

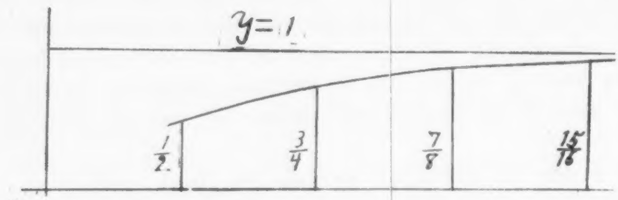

Figure 1

This curve approaches the line $y = 1$ as n increases. A similar situation exists for any convergent series, that is, if a convergent series $a_1 + a_2 + a_3 + \cdots$ has the sum s and its partial sums $s_1, s_2, s_3, \cdots$ are plotted as ordinates corresponding to the values 1, 2, 3, $\cdots$ of n and a smooth curve is drawn through these points, this curve approaches the line $y = s$ as n increases.

Now it is easy to see from the graph in Fig. 1 that the sequence of the arithmetic means of the first n ordinates approaches $y = 1$ as n increases. These means run $\frac{1}{2}, \frac{(\frac{1}{2}+\frac{3}{4})}{2}, \frac{(\frac{1}{2}+\frac{3}{4}+\frac{7}{8})}{3}, \cdots$ or $\frac{1}{2}, \frac{5}{8}, \frac{21}{24}, \cdots$. Similarly it is true for any convergent series that if s_n approaches s then $s_1, \frac{s_1+s_2}{2}, \frac{s_1+s_2+s_3}{3}, \cdots \frac{s_1+s_2+\cdots+s_n}{n}, \cdots$ also approaches s, in other words $\lim\limits_{n\to\infty} \frac{s_1+s_2+\cdots+s_n}{n} = s$. This can be seen analytically from the facts that the early ordinates do not count since $\lim\limits_{n\to\infty} \frac{s_1}{n} = \lim\limits_{n\to\infty} \frac{s_2}{n} = \cdots = \lim\limits_{n\to\infty} \frac{s_k}{n} = 0$ where k is any fixed constant, and all later terms can be made very nearly equal to s by taking k sufficiently large before fixing it.

Now it so happens that *the arithmetic mean of the first n partial sums of a divergent series may approach a limit.* For example, the partial sums of the series $1-1+1-1, \cdots$ being 1, 0, 1, 0, $\cdots$, the means of these partial sums are $1, \frac{1}{2}, \frac{2}{3}, \frac{1}{2}, \frac{3}{5}, \cdots$, or $1, \frac{1}{2}, \frac{1}{2}+\frac{1}{6}, \frac{1}{2}, \frac{1}{2}+\frac{1}{10}, \cdots$, the nth mean being $\frac{1}{2}$ when n is odd and $\frac{1}{2}+\frac{1}{2n}$ when n is even. Hence the limit of these means is $\frac{1}{2}$. Largely due to the work of Frobenius, this procedure of defining the limit of the arithmetic

1. There are other approaches but this is the type that we are interested in from the viewpoint of this paper.

mean of the first n partial sums of a divergent series as the sum of that series, if this limit exists, came to be accepted as a method of defining the *sum* of a divergent series.

There are, as one would naturally guess, numerous weighted means which, like the arithmetic mean, can be used to sum divergent series. Many definitions of the sum of divergent series have thus been formulated, all of which are regular. However they may not all give the same sum for a given divergent series even when they all sum it; *i.e.*, they may not be *consistent*.

The following paper concerns itself with the formulation of a general *regular* definition, all special forms of which are *consistent*. The most important result obtained is that the sum, by this general method, of any divergent series whose partial sums oscillate between fixed limits is the arithmetic mean of these limits, from which it follows that this is the sum of such series given by the various classic definitions of the sum of a divergent series.

Introduction. The general definition of the sum of a divergent series which is commonly called the Silverman-Toeplitz definition may be stated as follows:

If $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} a_i(n)s_i$ exists, where $s_i = \sum\limits_{j=1}^{i} a_j$, $a_i(n) = 0$, $i>n$,

(1) $\lim\limits_{n\to\infty} a_i(n) = 0$, for fixed i,

(2) $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} a_i(n) = 1$

(3) $\sum\limits_{i=1}^{n} |a\ (n)| < K$ independently of n,

then $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} a_i(n)s_i$ is the sum of the series Σa_i.

Silverman proved conditions (1), (2) and (3) sufficient to make this definition regular, in his doctorate disertation at the University of Missouri in 1910. Toeplitz[1] proved them necessary and sufficient, in 1911. But this definition is not (and probably was not intended to be) a means of mechanizing the actual summing of series because it does not define a class of *consistent* definitions[2]. We illustrate the latter point by means of a simple example, which also suggests the modification that this paper makes in the Silverman-Toeplitz definition with a view to securing consistency among its special cases.

Let $a_i(n)$ be h/n when n is even and $(2-h)/n$ when n is odd, where h is entirely arbitrary. These $a_i(n)$ are quite obviously a Silverman-Toeplitz transformation. Applying this transformation to the series

(1) Toeplitz, Prace Matematyzyno-fizycno, Vol. 22 (1911), p. 113.
(2) See Dienes, "The Taylor Series", pp. 390-391.

$1-1+1-1+\cdots$ we obtain $h/2$ for the sum of this series, which is correct only when $h = 1$[1]. The most outstanding distinction between the case in which h is 1 and any other value is that in the latter instances the sequences of a_i's oscillate, in other words change from monotonically decreasing (increasing) to monotonically increasing (decreasing), at every term except the first, while when h is 1 there are no such changes. This consideration suggests that we restrict the number of such changes of type of monotony. Letting $R_n - 1$ denote the number of changes of type of monotony in $a_1(n), a_2(n), a_3(n), \cdots a_n(n)$, we formulate the following definition[2]:

If $\lim_{n\to\infty} \sum_{i=1}^{n} a_i(n)S_i$ *exists, where* $S_i = \sum_{j=1}^{i} a_j$, $a_i(n) = 0$ when $i > n$,

(A) $R_n a_i(n) \to 0$ *uniformly with reference to i as n increases,*

(B) $\lim_{n\to\infty} \sum_{i=1}^{n} a_i(n) = 1$,

(C) $\sum_{i=1}^{n} |a_i(n)|$ *is bounded for all n,*

then the series Σa_i *is R-summable to* $\lim_{n\to\infty} \sum_{i=1}^{n} a_i(n)S_i$.

Any sequence satisfying (A), (B) and (C) will be called an R-semi-matrix transformation, or simply an R-matrix.

If a series is R-summable, it is obviously summable to the same sum by any method which satisfies not only the above assumptions but also any others consistent with these; and of course the latter methods may sum series which are not R-summable.

Certain intrinsic properties of the R-matrix will now be considered. For brevity a_i will be used to denote $a_i(n)$ unless the context requires the full notation.

Theorem I. *In any R-matrix,* $R_n = o(n)$, *i.e.,* R_n *is an infinity of lower order than n.*

Proof: According to hypothesis (A),

$$|a_1| < \frac{\epsilon}{R_n},\ |a_2| < \frac{\epsilon}{R_n},\ \cdots\ |a_n| < \frac{\epsilon}{R_n}, \qquad n > N_\epsilon .$$

Whence

$$(1) \qquad \sum_{i=1}^{n} |a_i| < \frac{n\epsilon}{R_n} \qquad n > N_\epsilon .$$

From hypothesis (B), we infer

$$(2) \qquad \sum_{i=1}^{n} a_i > 1 - \epsilon' \qquad n > N'_\epsilon, .$$

1. See Bromwich, "An Introduction to the Theory of Infinite Series", pp.260–265.
2. Presented to the Am. Math. Soc., Nov. 1942.

Combining (1) and (2)

$$\frac{n\epsilon}{R_n} > 1-\epsilon' \qquad\qquad n > N,\ N'.$$

From which

$$\frac{R_n}{n} < \epsilon + \epsilon' = \epsilon'',\ \text{since}\ \frac{R_n}{n} < 1.$$

This completes the proof.

Theorem II. *If $a_i(n)$ is an R-matrix transformation and m is any positive integer then $a_i(m+n)$, $i = 1, 2, 3, \cdots n$, is an R-matrix transformation.*

Proof: It suffices to prove that if $a_i(n+m)$ $i = 1, 2, 3, \cdots n+m$ is an R-matrix then the subsequence, $a_i'(n)$, $= a_i(n+m)$, $i = 1, 2, 3, \cdots n$, is also an R-matrix.

Now $R_{n+m} = R_n + R_m + C$, where C is either 0 or 1,

whence

$$R_{n+m} \geqq R_n$$

Then for $i \leq n$,

$$R_{n+m}\left|a_i(n+m)\right| \geqq R_n\left|a_n'(n)\right| = R_n\left|a_n(n+m)\right|.$$

By (A) the left member converges uniformly to zero on the range $0 < i \leqq n$, since it does for $0 < i \leqq n+m$.

Hence $R_n\left|a_i(m+n)\right|$ satisfies (A).

To show that

$$\lim_{n\to\infty} \sum_1^n a_i(n) = 1,$$

consider $\lim\limits_{n\to\infty} \sum\limits_1^{n+m} a_i(n+m)$, which is equivalent to $\lim\limits_{(n+m)\to\infty} \sum\limits_1^{n+m} a_i(n+m)$.

Write

$$\sum_1^{n+m} a_i(n+m) = \sum_1^n a_i(n+m) + \sum_{i=n+1}^{n+m} a_i(n+m).$$

The second summation on the right converges to zero as $n\to\infty$ by (A) since it contains only m terms. Thus (B) holds for $a_i'(n)$. Assumption (C) is, quite obviously, satisfied.

Theorem III. *If $a_j'(n)$, $j = 1, 2, 3, \cdots m$, is such that $\frac{1}{n} = o(a_j')$, (i.e., $\frac{1}{n}$ is an infinitesimal of higher order than a_j'), uniformly with reference to j, then*

$m = o(n)$

The proof follows very simply from the hypothesis when it is written

in the form

$$\frac{1}{n} < \epsilon |a_j'|, \text{ for all } j \text{ and } n > N_\epsilon .$$

For we can then write

$$\frac{m}{n} < \epsilon \sum_{j=1}^{m} |a_j'| < \epsilon K, \ n > N_\epsilon \text{ and } K \text{ a suitable positive constant.}$$

According to this theorem we might have R-matrices such as:

$$a_i(n) = \frac{1}{[n^{\frac{1}{a}}]}, \quad i \leq [n^{\frac{1}{a}}], \qquad a > 1$$

$$= 0 \quad , \quad i > [n^{\frac{1}{a}}], \qquad a > 1$$

Theorem IV. *If $a_j'(n)$, $j = 1, 2, 3, \cdots m$, be such that $a_j' = o(\frac{1}{n})$, uniformly with reference to j, then $(n-m) \to \infty$ whenever $n \to \infty$.*

Proof: Denote by a_k'' the a's which are not in the set a_j'. Then

$$(1) \quad \sum_1^n a_i = \sum_{j=1}^{m} a_j' + \sum_{k=1}^{n-m} a_k''$$

By hypothesis

$$(2) \quad \Big|\sum_{j=1}^{m} a_j'\Big| \leq \sum_{j=1}^{m} |a_j'| < \frac{m\epsilon}{n} < \epsilon \ , \qquad n > N_\epsilon \ .$$

Consequently

$$(3) \quad \lim_{n\to\infty} \sum_{k=1}^{n-m} a_k'' = \lim_{n\to\infty} \sum_1^n a_i = 1.$$

But if $n - m$ were bounded, then by (A)

$$\lim_{n\to\infty} \sum_{k=1}^{n-m} a_k'' = 0,$$

which contradicts (3).

As an illustration of this theorem, one might use $\frac{4}{n}$ for the first $[\frac{n}{4}]$ of the terms in (a_i) provided they are followed by terms of higher order than $\frac{1}{n}$, say $\frac{1}{n!}$. However these two classes of terms cannot be indiscriminantly mixed on account of (A).

The following theorem makes possible the actual summing of certain divergent series by means of *any* R-matrix transformation.

Theorem V. *If $\lim_{n\to\infty} R_n a_i = 0$ uniformly with reference to i, where R_n is one greater than the number of changes of monotony in (a_i), and if c is any integer and m is the greatest integer in $\frac{n}{c}$, then*

$$\lim_{n\to\infty} \sum_{i=1}^{n} a_i = c \lim_{n\to\infty} \sum_{i=1}^{m} a_{c+c(i-1)}$$

if either limit exists.

We prove the theorem for $e = c$. Other values of e only modify the notation.

Proof: Since changing the signs of all terms in (a_i) changes all of the types of monotony, it suffices to consider only sequences which begin with a set of monotonically decreasing terms.

If the number of terms in this first set is not an exact multiple of c, we replace enough of the succeeding a's by the last a in the first set to make the number of terms in the modified set an exact multiple of c, say $m_1 c$. In doing that at most $c-1$ interchanges of a's will have been made. Assuming that there remains a set of monotonically increasing terms, we give it the same treatment and denote its modified form by $m_2 c$. After this treatment has been applied to all but the last set in (a_i), we delete enough of the last terms in this last set to make the number of terms remaining in it an exact multiple of c and denote this modified last set by $m_R c$. The entire new sequence of a's, we denote by $a'_i(n')$, $i = 1, 2, 3, \cdots n'$, where $n' = [\frac{n}{c}]c$.

Now let a_h be the largest a replaced or deleted and a_k the smallest a which replaces any a, in the above transformation, and note that at most $(R_n - 1)$ sets have been modified by substitutions. Then we can formulate the following inequalities.

$$(1)\quad |(R_n-1)\,c\,(a_h-a_k)| + |(c-1)a_h| \;\overline{>}\; \sum_{i=1}^{n} a_i(n) - \sum_{i=1}^{n'} a'_i(n')$$

$$\overline{>}\; -\,|(R_n-1)\,c\,(a_h-a_k)| - |(c-1)\,a_h|\,, \text{ where } n' = [\tfrac{n}{c}]c,$$

and

$$(2)\quad |R_n(c-1)(a_h-a_k)| \;\overline{>}\; \sum_{i=1}^{m} a_{ci}(n) - \sum_{i=1}^{m'} a'_{ci}(n') \;\overline{>}\; -\,|R_n(c-1)(a_h-a_k)|,$$

where $m = [\frac{n}{c}]c$ and $m' = [\frac{n'}{c}]c$.

The first and last members of each of these inequalities approaches zero as n increases, because of (A). Hence it suffices to prove our theorem for $\sum_{i=1}^{n'} a'(n')$, where $n' = [\frac{n}{c}]c$.

Considering sets $m_1 c$ and $m_2 c$, we have the following inequalities, in which a has been written for a'.

For set $m_1 c$

$a_1 = a_1 \geq a_c$,

$a_1 \overline{>} a_2 \geq a_c$,

. . . .

$a_1 \overline{>} a_c = a_c$,

$a_c \overline{>} a_{c+1} \overline{>} a_{2c}$,

$a_c \overline{>} a_{c+2} \overline{>} a_{2c}$,

. . . .

For set $m_2 c$

$a_{(m_1+1)c} \overline{>} a_{m_1 c+1} \overline{>} a_{m_1 c}$,

$a_{(m_1+1)c} \overline{>} a_{m_1 c+2} \overline{>} a_{m_1 c}$,

. . . .

$a_{(m_1+1)c} = a_{(m_1+1)c} \overline{>} a_{m_1 c}$,

$a_{(m_1+2)c} \overline{>} a_{(m_1+1)c+1} \overline{>} a_{(m_1+1)c}$,

$a_{(m_1+2)c} \overline{>} a_{(m_1+1)c+2} \overline{>} a_{(m_1+1)c}$,

. . . .

$$\begin{array}{ll}
a_c \geq a_{2c} = a_{2c}, & a_{(m_1+2)c} = a_{(m_1+2)c} \geq a_{(m_1+1)c}, \\
\cdots\cdots\cdots & \cdots\cdots\cdots\cdots \\
a_{(m_1-1)c} \geq a_{(m_1-1)c+1} \geq a_{m_1c}, & a_{(m_1+m_2)c} \geq a_{(m_1+m_2-1)c+1} \geq a_{(m_1+m_2-1)c}, \\
a_{(m_1-1)c} \geq a_{(m_1-1)c+2} \geq a_{m_1c}, & a_{(m_1+m_2)c} \geq a_{(m_1+m_2-1)c+2} \geq a_{(m_1+m_2-1)c}, \\
\cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots \\
a_{(m_1-1)c} \geq a_{m_1c} = a_{m_1c}, & a_{(m_1+m_2)c} = a_{(m_1+m_2)c} \geq a_{(m_1+m_2-1)c}, \\
-c\sum_{i=1}^{m_1} a_{ci} = -c\sum_{i=1}^{m_1} a_{ci} = -c\sum_{i=1}^{m_1} a_{ci}, & -c\sum_{m_1+1}^{m_1+m_2} a_{ci} = -c\sum_{m_1+1}^{m_1+m_2} a_{ci} = -c\sum_{m_1+1}^{m_1+m_2} a_{ci},
\end{array}$$

Adding each of these two sets gives,

$$c_1a_1 - ca_{m_1} \geq \sum_1^{m_1c} a_i - c\sum_1^{m_1} a_{ci} \geq 0 \quad 0 \geq \sum_{m_1c+1}^{(m_1+m_2)c} a_i - c\sum_{m_1+1}^{m_1+m_2} a_{ci} \geq ca_{m_1} - ca_{(m_1+m_2)c}$$

The total number of sets, m_1c, m_2c, $\cdots$ m_Rc, is at most R_n. Hence when we add all such inequalities as the above, we obtain for the first member a quantity not greater than $2cR_n|a_g|$ where a_g is the numerically largest $a_i(n')$, and for the last member a quantity not less than $-2cR_n|a_g|$. Thus we have

$$2cR_n|a_g'| \geq \sum_1^{n'} a_i'(n') - c\sum_1^{m'} a_{ci}'(n') \geq -2cR|a_g'(n')|, \quad m' = \frac{n'}{c}.$$

The conclusion of the theorem follows easily from this inequality, inequalities (1) and (2) and hypothesis (A).

Corollary 1. *If* (a_i) *is an R-matrix so is* (ca_{ci}), $i = 1, 2, 3, \cdots m$, *where* $m = [\frac{n}{c}]$.

Corollary 2. *If* (a_i), $i = 1, 2, \cdots n$, *is an R-matrix, so is* $\dfrac{c_1a_i + c_2a_{i+1} + \cdots c_ka_{i+k}}{c_1 + c_2 + \cdots c_k}$, $i = 1, 2, 3, \cdots m$, *where the c's are such that* $\sum_1^k c_j \neq 0$ *and* $m = [n/\sum_1^k c_j]$.

We are now in a position to establish some useful theorems concerning the sums of certain divergent series. For brevity we shall denote $\lim_{n\to\infty} \sum_1^n a_iS_i$, under conditions (A), (B) and (C) by $R\Sigma a_i$, where $S_i = \sum_{j=1}^{i} a_j$.

Theorem VI. *If the partial sums of* Σa_i *oscillate between finite numbers then* $R\Sigma a_i$ *is the arithmetic mean of these numbers, i.e., the R sums of a series whose partial sums are* A_1, A_2, $\cdots$ A_k, A_1, A_2,

$\cdot\cdot A_k, \cdots$ is $\dfrac{A_1 + A_2 + \cdots + A_k}{k}$.

Proof: The sum $\sum_{i=1}^{n} a_i S_i$ differs by $n - [n/k]k$ terms from

$$\sum_{i=1}^{m} a_{1+k(i-1)} A_1 + \sum_{i=1}^{m} a_{2+k(i-1)} A_2 + \cdots + \sum_{i=1}^{m} a_{k+k(i-1)} A_k, \quad m = [n/k].$$

Hence by (A) and theorem V, we have,

$$\lim_{n\to\infty} \sum_{i=1}^{n} a_i S_i = \lim_{n\to\infty} \sum_{i=1}^{m} a_{1+k(i-1)} A_1 + \lim_{n\to\infty} \sum_{i=1}^{m} a_{2+k(i-1)} A_2 + \cdots +$$

$$\sum_{i=1}^{m} a_{k+k(i-1)} A_k = \frac{A_1}{k} + \frac{A_2}{k} + \cdots + \frac{A_k}{k}.$$

because

$(S_i) = 1, 2, 3, 1, 2, 3, \cdots$.

2) $R\,(1 - 1 + 1 \cdots) = \frac{1}{2}$, as it should in order to be consistent with classic usuage.

The methods of Euler, Cesaro, Hölder, Le Roy and the other classic methods which transform sequence into sequence are all R-matrix transformations. Hence they give the above sums for series whose partial sums oscillate between constant numbers. Borel's integral definition, also, gives the same sum. For whenever Cesaro's method sums a series Borel's integral definition sums it to the same sum.

Corollary. If the sequence of partial sums of a divergent series can be broken up into subsequences each of which belongs to an R-summable series, then the original series is R-summable to the mean of the sums of the latter series.

In particular if a sequence consists of subsequences each of which converges, then the sum of the series corresponding to the original sequence is the arithmetic mean of the limits of these subsequences.

Theorem VII. *If the series whose sequence of partial sums is* (S_i) *is R-summable to S, then the series whose sequence of partial sums is* $\left(\sum_{j=1}^{m} c_j S_{j+i} a_i\right)$, *m constant and* $\sum_{j=1}^{m} c_j = 1$, *is R-summable to S.*

To prove this theorem, construct the following m sets of R matrices,

$$a_{i'}(n+k-k) \begin{cases} = 0, & i' \leq k, \quad k < i' \leqq n+k \\ = a_{i'}(n), & i' > k, \quad k = 1, 2, 3, \cdots m. \end{cases}$$

By hypothesis,

$$S - \epsilon_1 < S_1 a_1 + \cdots + S_n a_n < S + \epsilon_1, \quad n > N_{\epsilon_1}$$
$$S - \epsilon_2 < S_2 a_1 + \cdots + S_n a_n < S + \epsilon_2, \quad n > N_{\epsilon_2},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$S-\epsilon_m < S_m a_1 + \cdot \cdot \cdot + S_n a_n < S+\epsilon_m, \quad n > N_{\epsilon_m}.$$

Multiplying these inequalities, in order, by $c_1, c_2, \cdot \cdot \cdot c_m$, letting ϵ' be the largest of the ϵ's and making use of $\sum_{j=1}^{m} c_j = 1$, we can infer

$$S-\epsilon' < \sum_{i=1}^{n} \Big(\sum_{j=1}^{m} c_j S_{j+i} \Big) a_i < S+\epsilon', \quad N > N_{\epsilon_1}, N_{\epsilon_2}, \cdot \cdot \cdot N_{\epsilon_m}.$$

Whence

$$\lim_{n\to\infty} \sum_{i=1}^{n} \Big(\sum_{j=1}^{m} c_j S_{j+i} \Big) a_i = S.$$

The converse of this theorem is not true, as can be seen from the counter example in which the sequence of partial sums is

$$2, -2, 3, -3, \cdot \cdot \cdot .$$

Taking the means two at a time gives

$$0, \tfrac{1}{2}, 0, \tfrac{1}{2}, \cdot \cdot \cdot \cdot$$

which sums to $\frac{1}{4}$, whereas the original series is not R-summable because of theorem VIII.

Theorem VII seems to be most useful in showing that certain series are not R-summable, *e.g.*, if the partial sums of a series are, 1, −1, 2, −1, 3, −1, $\cdot \cdot \cdot$ the series is not R-summable, for $\left(\frac{S_i + S_{i+1}}{2}\right)$ is properly divergent, therefor not summable by $a_i = 1/n$.

It is a matter of very elementary limit theory to show that the properties which we customarily investigate for definitions of summability hold for R-summability. *E.g.* 1) If each of two series is R-summable, their term by term sum is R-summable to the sum of the R sums of the two series, but not conversely. 2) If Σa_i is R-summable to S, then $\Sigma\, a_{i+1}$ is R-summable to $R\, \Sigma\, a_i - a_1$ and conversely.

Theorem VIII. *A necessary condition for R-summability is that*

$$\lim_{n\to\infty} S_n a_n = 0$$

This is a consequence of theorem II. For by that theorem

$$\lim_{n\to\infty} \sum_{i=1}^{n+1} S_i a(n+1) - \lim_{n\to\infty} \sum_{i=1}^{n} S_i a_i (n+1) = 0$$

whenever the first term exists.

This restriction and theorem III show that a series whose n^{th} partial sum is unbounded is not R-summable. For, whatever unbounded form S_n takes on one can certainly construct an R-matrix with terminal a's such that $\lim_{n\to\infty} S_n a_n \neq 0$. On the other hand, given an S_n one can construct a special R-matrix such that theorem VIII holds. (But this, of course, does not guarantee summability.) Along this line it is an interesting, and open, question as to whether any special R-matrix can sum any properly divergent series.

University of California

ON EXTENSORS AND THE LAGRANGIAN EQUATIONS OF MOTION

Homer V. Craig

FOREWORD

The underlying problem on which this paper bears is the following: Given the position and velocity of a particle at a certain instant together with a specification of the forces acting during its flight, determine a set of formulas which will successfully predict the position and, indirectly, the velocity of the particle at all times during the motion. The *position* of the particle is given by its *coordinates*, while its *velocity* is determined by the *time rate of change of these coordinates*.

By the term coordinates of a point or particle, we mean a set of numbers which will suffice to locate or identify the particle according to some prearranged scheme. Obviously, there are many schemes (infinitely many as a matter of fact) for assigning sets of numbers to points. One scheme, for example, would be to use the latitude, longitude, and distance above the earth. Or it might be more convenient to locate a point by saying that it is x feet east of a given base point O, y feet north, and z feet up. Obviously, a mere shift in the base point O would give a new system for attaching number triplets x, y, z to points. Still other systems could be obtained by defining new variables $\overline{x}$, $\overline{y}$, $\overline{z}$ by means of equations relating them to a previous set x, y, z. To sum up, a coordinate system is merely a scheme for attaching identifying numbers to points and there is an endless variety of such schemes. We introduce a coordinate system into the study of a given problem to help us formulate the question in a precise numerical form. For example, the basic problem underlying our paper is to determine each of the coordinate variables x, y, z belonging to the moving particle as an expression in t. For example, the answer to a specific problem of this type might be $x = 2000t$, $y = 0$, $z = -16.1t^2 + 1000t$. Here t is the time in seconds measured from a certain particular instant. In order to determine such a specific expression for the behavior of a particle, we must have at our disposal: (1) a coordinate system, (2) a formulation of the forces which act on the particle during its flight, (3) the universal law which governs the motion of particles, and (4) methods for extracting from the universal law the equations which express explicitly the coordinate variables of the particle in the specific problem at hand in terms of the time.

With regard to these requisites, surely the most fascinating is the third. What is the universal law governing the motion of particles? In the final analysis, the determination of such a law must rest to some extent on guess work. The investigator supposedly has certain facts of observation at his disposal and these facts may lead directly

to a tentative formulation of the desired law or they may suggest some general assumptions from which the law may be deduced. Of course, after a tentative universal law has been formulated it may be tested partially by comparing suitable deductions from the law with pertinent experimental facts. For example, if following Newton, we assume that the universal law for particles asserts the equality of the force acting on the particle and the time rate of change of momentum (momentum is mass times velocity), then we may solve a variety of particular problems on the basis of this universal law and compare predictions based on the solutions with the corresponding experimental observations.

The purpose of our paper is to present a set of assumptions which will lead easily to a certain very useful form (called the Lagrangian equations) of Newton's fundamental law relating the force acting on the particle to the rate of change of the momentum of the particle. Our main assumption is in keeping with a well known principle for the construction of tentative formulations of fundamental laws. This principle derives from the artificial and extraneous status of a coordinate system that has been introduced into the study of a physical problem. To clarify this matter it will be well for us to reexamine our underlying problem.

To fix the ideas let us suppose that the particle whose path we are to determine is a projectile fired into a vacuum and subject to no forces while in flight excepting the force of gravitation. Obviously, the actual behavior of this particle as contrasted to the x, y, z, t-description of this behavior is independent of the particular coordinate system which is introduced to motivate the analysis. For example, the actual point at which the particle ends its flight is independent of the coordinate system used although the *coordinates* of the point, *i.e.* the number triplet associated with the point, do depend on the system introduced. This obvious independence of physical behavior on the coordinate system introduced as an aid to ascertaining and describing that behavior leads to the following conjecture. It should be possible to construct a symbolism by means of which the laws of physics would be expressible in a form that is valid in all coordinate systems and, more important, the invariantive properties of the new formalism should be useful as an aid in determining or guessing the fundamental laws.

About the turn of the century two Italian mathematicians M.G.M. Ricci and Tullio Levi-Civita succeeded in constructing and developing such a formalism. This particular branch of mathematics is known variously as the *absolute differential calculus* or *tensor analysis*. Roughly fifteen years later Einstein employed this new branch of mathematics in the development of his celebrated general theory of relativity, and at various times numerous other investigators have found it fruitful to employ similar procedures.

A certain natural extension of the basic entity of tensor analysis is the concept extensor and our principle assumption is merely that

the basic general law governing the motion of particles is expressible as an equation in extensors or, more specifically, in the extensors that can be constructed by computing rates of change of the energies of motion and position taken separately.

1. *Introduction.* Most developments of the Lagrangian equations for the motion of a particle start with the Newtonian equations as the primary datum and then proceed either directly or through the development of new principles to effect a solution of the inverse problem of the calculus of variations. An alternative procedure is to set up the Euler-Lagrange equations associated with the kinetic-potential and show that they are equivalent to Newton's equations. In the present note we propose to attack the problem on the basis of somewhat lighter assumptions. We shall consider neither the kinetic-potential nor the Newtonian equations to be given but adopt instead the hypothesis that the equations of motion are expressible as linear equations in the *primary extensors* associated with the kinetic and potential energies separately. The *primary extensors* associated with a given function are those that are obtainable by differentiation alone (partial or total) without subsequent combination, hence they represent pure rates of change. In order to present this concept explicitly and to pave the way for our argument, it will be well to review briefly certain elements of the theory of extensors.

2. *Properties of Extensors.* The extensors which we shall employ follow one or the other of the two transformation equations

$$(2.1) \qquad E_{aa} = \overline{E}_{\rho r} X^{\rho r}_{aa}, \qquad (2.2) \qquad E^{aa} = \overline{E}^{\rho r} X^{aa}_{\rho r}.$$

Here the E's are the components of the extensor while the X's are the partial derivatives associated with the extended coordinate transformation, specifically, X sup ρr inf $aa = \partial \overline{x}^{(\rho)r}/\partial x^{(a)a}$, $x^{(a)a} = d^a x/dt^a$. The range of the Latin indices is 1 to N (N is the dimensionality of the space), while the range of the Greek letters is 0 to M. Repeated lower case indices are summed over their ranges, while capital indices do *not* generate sums. Some of the special properties of extentors derived from the fact that X sup ρr inf aa is equal to the product of the binomial coefficient ${}_\rho C_a$ and $(\partial \overline{x}^r/\partial x^a)^{(\rho - a)}$. In particular, the foregoing X vanishes whenever the Greek subscript exceeds the Greek superscript and reduces to $\partial \overline{x}^r/\partial x^a$ if the Greek indices are equal.

Simple algebraic properties. As in tensor analysis a linear combination of two extensors of the same type, with scalar coefficients is again an extensor of that type. Also, if all of the components of an extensor in any one coordinate system vanish then the same may be asserted for the other coordinate systems. An immediate consequence

is the paramount conclusion that a linear extensor equation is valid in all admissible coordinate systems if it is valid in any one. In a word, *extensor equations are invariant equations*. Furthermore, examination of the equations (2.1), (2.2) will show that extensor components fall naturally into sets according to the value of the Greek indices. We shall call these sets *ranks* and see that they possess a certain interesting property.

The independence property of ranks. Briefly, the independence property simply asserts that the extensor components in a coordinate system x which are of specified rank depend only on the components in other systems of equal or more active rank. In case of an extensor of the type $E_{\alpha a}$, the larger the value of the subscript α the more active the rank. In particular, the components E_{Ma} (associated with the system x) depend only on the components $\bar{E}_{\rho r}$ (of system $\bar{x}$) having $\rho = M$. These components of most active rank constitute the components of a tensor. In the case of an extensor which bears a Greek superscript, the smaller the value of the superscript the more active the rank. The most active rank in this case (Greek superscript zero) is again the tensor rank. The independence property is an immediate consequence of the transformation equations (2.1), (2.2) and the properties of the coefficients X. The tensor components are the most active of all in the sense that tensor components appear in the right members of (2.1) and (2.2) regardless of the value of α. Closely associated with the idea of rank and the independence property is a rule relating to advancement in rank.

The promotion rule. The essence of this rule is that if all of the components in any one coordinate system which are of more active rank than the rank which is R ranks removed from tensor rank vanish, then these zero components may be discarded and the rank R-removed promoted to tensor rank. To illustrate, if M is unity and $\bar{E}_{1r} = 0$, then $E_{1a} = 0$ and E_{0a} (the rank 1-removed from tensor rank) may be promoted to the tensor rank since E_{0a} is now given by $E_{0a} = \bar{E}_{0r}\partial\bar{x}^r/\partial x^a$. The value of M may now be lowered if desired. This possibility of discarding ranks bears on the question of the *degree of determination* imposed by extensor equations.

The degree of determination. Since an extensor equation may impose as many as $(M+1)N$ conditions on the quantities to be determined, assuming that it is of the type which we are considering, the question of over determination may arise. An instance is furnished when the equations of motion of a particle are presented in extensor form. Here M is unity and there are $2N$ equations to determine the N coordinates as functions of the time. Because of the possibility of linear dependence the equations might be just sufficient to determine the functions except for constants of integration, or they might exclude certain orbits otherwise possible, or the entire set might be inconsistent.

3. *Some simple examples of extensors*. The extensors which we shall take as fundamental in the present paper are the *primary extensors* associated with the important scalars. These extensors, as stated in the introduction, are made up of rates of change. Specifically, if f is a scalar function of the coordinate variables and their derivatives with respect to t up to and including at most derivatives of order M, then the primary extensors* associated with f, namely, $f_{;\alpha a}$ and $f_{\alpha a}$ may be defined as follows

$$(3.1) \quad f_{;\alpha a} = \partial f/\partial x^{(\alpha)a} \quad ; \quad f_{\alpha a} = {}_M C_A f_a^{(M-A)}$$

$$(A = \alpha \text{ but is not summed}).$$

Here f_a is a *tensor included in the set* $f_{;\alpha a}$ *which does not vanish identically.*

To illustrate this definition, let us take M to be unity and consider the kinetic energy T and the potential energy U – the former is a function of the x's and x''s, while the latter depends on the x's alone. The components of $T_{;\alpha a}$, arranged in ranks are $T_{;1a}$ and $T_{;0a}$, while those of $U_{;\alpha a}$ are $U_{;1a}(=0)$ and $U_{;0a}$. $T_{;1a}$ is a tensor which presumably does not vanish identically, and hence $T_a = T_{;1a}$. On the other hand, since the tensor $U_{;1a}$ is identically zero, we must turn to the tensor $U_{;0a}$ as the only choice for U_a. Consequently, since the binomial coefficients in the present case ($M = 1$) all have the value unity, the extensors $T_{\alpha a}$ and $U_{\alpha a}$ are given by

$$(3.3) \quad T_{1a} = T_a = T_{;1a} \; ; \quad T_{0a} = T_a{}' = T_{;1a}{}' \; ;$$

$$(3.4) \quad U_{1a} = U_a = U_{;0a} \; ; \quad U_{0a} = U_a{}' = U_{;0a}{}' \; .$$

The linear extensor equation associated with $f(x,x')$. As an application of certain of the facts just presented let us assume that we have given a function $f(x,x')$ of the N coordinate variables and their first derivatives with respect to a parameter t and examine the linear equation in the associated primary extensors: $f_{;\alpha a}$ and $f_{\alpha a}$. We shall write this equation in the form $f_{\alpha a} + Af_{;\alpha a} = 0$. Expanding into ranks and evaluating $f_{\alpha a}$ by means of formulas of the type (3.3), we obtain

$$(3.5) \quad f_{;1a} + Af_{;1a} = 0 \; ; \quad f_{;1a}{}' + Af_{;0a} = 0.$$

Here we have N first order and N second order differential equations in the N coordinates x. Consequently, the question of over determination is a pressing one here. The simplest way to settle it is to give A the value minus one, since this will eliminate the tensor group (subscript

*For proofs of the extensor character of these quantities see items 1, 4 and 5 of the appended bibliography.

1) in all coordinate systems and secure the promotion of the other group to tensor rank. The result is the tensor equation $df_{;1a}/dt - f_{;0a} = 0$, which we recognize as the set of Euler equations associated with the calculus of variations problem corresponding to $\int f\, dt$.

It may be worth noting here that the geodesic equations $x''^a + \{{}_b{}^a{}_c\}x'^b x'^c = 0$ admit of a similar structural analysis. The extensors involved are the motion extensor for $M = 1$, namely, x'^a, x''^a and the connection extensor δ^a_b, $\{{}_b{}^a{}_c\}x'^c$ contracted with x'^b.

Before turning to the development of the equations of motion, we should note that the extensor components presented in (3.3) and (3.4) are of primary importance in physics since they consist of momenta, rates of change of momenta, and the force or rather its negative. For this reason and because extensors include tensors among their components and thus present a broader basis for forming equations, we shall study our problem from the standpoint of extensors. In particular we shall not deliberately introduce the non-primary tensor $df_{;1a}/dt - f_{;0a}$

4. *An heuristic approach to the kinetic-potential.* To obtain tentative equations of motion, we make the following assumptions:

(a) The important functions are the kinetic energy $T(x,x')$, which is homogeneous of degree two in the x''s, and the potential energy $U(x)$; and, specifically, a linear relationship among the primary extensors associated with these functions holds along a trajectory;

(b) The total energy $T + U$ is constant along a trajectory;

(c) The extensor component of functional order two, namely, T_{0a} must be present (this excludes the possibility of having only first order differential equations).

By virtue of (c), we may take the coefficient of T_{aa} in the relationship predicated in (a) to be unity and write

$$(4.1)\quad T_{aa} + AT_{;aa} + BU_{aa} + CU_{;aa} = 0,$$

or expanding into ranks and employing the relationships (3.3) and (3.4)

$$(4.2)\quad T_{;1a} + AT_{;1a} + BU_{;0a} + C \cdot 0 = 0,$$

$$(4.3)\quad T_{;1a}' + AT_{;0a} + BU_{;0a}' + CU_{;0a} = 0.$$

In order to prevent over determination, we take $A = -1$, $B = 0$. This renders (4.2) an identity and promotes (4.3) to tensor status. Thus we have

$$(4.4)\quad T_{;1a}' - T_{;0a} + CU_{;0a} = 0.$$

In order to evaluate C, we multiply (4.4) by x'^a and note that the homogeneity of T implies that $x'^a T_{;1a}' = 2T' - x''^a T_{;1a}$. Thus, we have $2T' - x''^a T_{;1a} - x'^a T_{;0a} + Cx'^a U_{;0a} = 0$ or $T' = -CU'$. But assumption (b) asserts that $T' = -U'$, hence C must be unity along a trajectory at

least and (4.4) reduces to the Euler equations associated with the kinetic-potential.

Perhaps it should be stated in conclusion, that although we have reached the same final equations as in the usual treatments of the subject, the emphasis is no longer on the kinetic-potential and the Eulerian operator $d/dt\,\partial/\partial x'^{a} - \partial/\partial x^{a}$ but instead is on the kinetic and potential energies *separately* and their *primary extensors* which consist of quantities that are otherwise of fundamental importance in mechanics and, in addition, are expressible as pure rates of change. Furthermore, it should be noted that the invariants A, B, C need take on the assigned values only along the trajectories, and their presence presents the possibility of excluding certain orbits if desired.

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University of Texas

CURRENT PAPERS AND BOOKS

Edited by

H. V. Craig

This department will present comments on papers previously published in the MATHEMATICS MAGAZINE. lists of new books, and book reviews.

In order that errors may be corrected, results extended, and interesting aspects further illuminated, comments on published papers in all departments are invited.

Communications intended for this department should be sent in duplicate to H. V. Craig, Department of Applied Mathematics, University of Texas, Austin 12, Texas.

Reply to A. M. Mood's Review in the March-April Mathematics Magazine.

As we stated in the preface to our *Fundamentals of Statistics*, the book was intended for use in a very brief course (about 20 recitations) in statistics for students who had the mathematical background represented by the usual first course in calculus. The book contains more material than could be covered in such a brief course, but we felt that the extra material should be included for the sake of completeness. This restriction of time and space was the chief factor in our consideration of what to include and what to leave out.

Two other factors had considerable influence on our decisions in connection with this choice. The first was a resolution to avoid, in so far as was consistent with the restrictions of space and the capabilities of the students, the use of results which could not be derived or, at least, made plausible. The second was a desire to avoid over-emphasis of the normal distribution. Its importance stems from its appearance in the conclusion of the central limit theorem, and we tried to keep it in this light by minimizing its use as a part of descriptive statistics.

We decided to omit the small sample distributions and the techniques based on them for a variety of reasons, but primarily because of the restriction of space. The strict applicability of the small sample distributions is in many cases contingent upon the normality of the distribution of an underlying variable. Thus, although they provide powerful tools, the advisability of including them in a quick, first look at statistics is questionable. Moreover, a hypothesis is tested in essentially the same way whether the normal distribution or a small sample distribution provides the proper probability scale.

Dr. Mood states that statisticians buried skewness and kurtosis years ago. If so, it seems that some of the leaders have since decided to resurrect them; for it is to be noted that when Kendall took over the revision of the long- and widely-used Yule he inserted a chapter on these topics in his first revision[1], and he later included a short treatment of them in his own monumental work.[2] Cramer, too, gives a brief discussion of skewness and kurtosis in his recent classical treatise.[3] Our desire to de-emphasize the normal distribution as a basic

tool of descriptive statistics obligated us to present an alternative. We chose the Gram-Charlier system of curves on the basis of their simplicity. This system, or any system fitted by equating moments, necessitates some discussion of the moments of a frequency distribution. Consequently we felt that these topics should be included.

The book does have some weaknesses, but they are due mostly to the enforced brevity of treatment and to the over-simplification which results therefrom. To get a good look at a landscape, some underbrush must be cleared away. We tried to show what statistics is about; maybe we failed.

U. S. Naval Academy — J. B. Scarborough
Oberlin College — R. W. Wagner

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COLLEGIATE ARTICLES

Graduate training not required for Reading

AN APPROACH TO NON-EUCLIDEAN TRIGONOMETRY

Curtis M. Fulton

It is the purpose of this paper to derive the basic trigonometric formulas of hyperbolic and elliptic geometry from a suitable system of axioms. Such a system, common for Euclidean, hyperbolic and elliptic geometry, has to consist of (1) axioms of incidence, (2) separation, (3) continuity and (4) congruence. We shall omit an explicit statement of these axioms, because we shall need just a few simple geometric facts, which we take for granted.

Let α, β, γ denote the angles opposite the sides a, b, c of a triangle, respectively. It is convenient to anticipate the following

LEMMA 1. *In any triangle*

$$\beta + \gamma \neq \pi + \alpha. \tag{1}$$

If $\beta < \alpha$ or $\gamma < \alpha$ there is nothing to prove. Thus, we assume that $\gamma > \alpha$

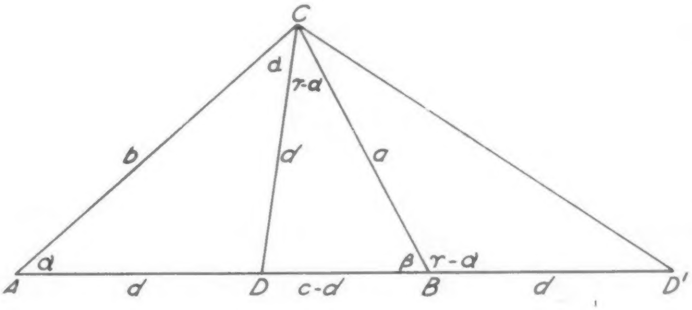

Figure 1

and make $\angle ACD = \alpha$ (Fig. 1). Consequently $AD = DC = d$. Let us assume now that our proposition is not true, in which case $\gamma - \alpha = \pi - \beta$. We lay off $BD' = d$ and have according to an axiom of congruence[1]: $\angle BCD' = \angle CBD = \beta$. Hence D' would lie on DC and at the same time on the line AB, *i.e.* D' would coincide with D; or, in other words, the segment DD', for which B is an interior point, would be equal to the entire length of the straight line AB. But on the other hand the

1. See e.g. R. Baldus, "Nichteuklidische Geometrie", Berlin und Leipzig, 1927, p. 31.

distance $DD' = DB + BD' = c - d + d = c$, whence it follows that DD' cannot equal the entire line. As our assumption leads to a contradiction, the lemma is true.

We introduce now a function of the angles of a triangle, that characterizes their sum as being greater or less than π, respectively. For this we make use of the fact that cosine is a decreasing function for arguments from 0 to π. If we assume first that $\alpha + \beta + \gamma < \pi$, it follows that $\alpha + \beta < \pi - \gamma$ and $\cos(\alpha + \beta) > -\cos\gamma$ or

$$(2) \qquad \frac{\cos\gamma + \cos\alpha\cos\beta}{\sin\alpha\sin\beta} > 1.$$

If, on the other hand $\alpha + \beta + \gamma > \pi$ and $\alpha + \beta < \pi$, we obtain in a similar fashion

$$(3) \qquad \frac{\cos\gamma + \cos\alpha\cos\beta}{\sin\alpha\sin\beta} < 1.$$

In the case of $\alpha + \beta > \pi$, there is no restriction of the preceding type on γ. Finally, if $\alpha + \beta + \gamma = \pi$, our function becomes equal to 1.

Before we investigate the nature of the function

$$\frac{\cos\gamma + \cos\alpha\cos\beta}{\sin\alpha\sin\beta}$$

we can show that, excluding the case $\alpha + \beta + \gamma = \pi$ and considering (1), this function cannot be equal to 1 or -1. For, suppose that

$$\frac{\cos\gamma + \cos\alpha\cos\beta}{\sin\alpha\sin\beta} = 1$$

whence we get $\cos(\alpha + \beta) = -\cos\gamma$. Since all the angles in question are angles between 0 and π, we can infer that

$$\alpha + \beta = \pi - \gamma \quad \text{or} \quad \alpha + \beta = \pi + \gamma,$$

both relations having been ruled out. If, in turn, we assume that

$$\frac{\cos\gamma + \cos\alpha\cos\beta}{\sin\alpha\sin\beta} = -1,$$

we have $\cos(\alpha - \beta) = -\cos\gamma$. Hence, for the possible range of values of these angles,

$$\alpha - \beta = \pi - \gamma \quad \text{or} \quad \alpha - \beta = \gamma - \pi.$$

Both relations are impossible according to (1). Thus,

$$(4) \qquad 1 - \left[\frac{\cos\gamma + \cos\alpha\cos\beta}{\sin\alpha\sin\beta}\right]^2 \neq 0.$$

According to a theorem of congruence, a triangle is determined by its sides, so that the angles will be well defined functions of the sides, and the same is true for any given function of these angles. Thus we may set:

$$(5)\qquad \frac{\cos\alpha+\cos\beta\cos\gamma}{\sin\beta\sin\gamma}=F(a,b,c)\qquad \frac{\cos\beta+\cos\gamma\cos\alpha}{\sin\gamma\sin\alpha}=F(b,c,a)$$

$$\frac{\cos\gamma+\cos\alpha\cos\beta}{\sin\alpha\sin\beta}=F(c,a,b),$$

where F stands for a certain unknown function. On account of the symmetry we clearly have:

$$F(a,b,c)=F(a,c,b).$$

We first apply these functions to a right triangle with $\gamma=\frac{\pi}{2}$ and have

$$\frac{\cos\alpha}{\sin\beta}=F(a,b,c)\qquad \frac{\cos\beta}{\sin\alpha}=F(b,c,a)\qquad \frac{\cos\alpha\cos\beta}{\sin\alpha\sin\beta}=F(c,a,b).$$

Hence,

$$(6)\qquad F(a,b,c)F(b,c,a)=F(c,a,b).$$

This equation expresses the fact that in a right triangle the three sides are not independent.

By means of a simple computation (5) yields

$$1-F^2(a,b,c)=\frac{1-\cos^2\alpha-\cos^2\beta-\cos^2\gamma-2\cos\alpha\cos\beta\cos\gamma}{\sin^2\beta\sin^2\gamma}$$

whereupon we infer immediately the "law of sines"

$$(7)\qquad \frac{1-F^2(a,b,c)}{\sin^2\alpha}=\frac{1-F^2(b,c,a)}{\sin^2\beta}=\frac{1-F^2(c,a,b)}{\sin^2\gamma}$$

Also,

$$F(c,a,b)-F(a,b,c)F(b,c,a)=\cos\gamma\,\frac{1-\cos^2\alpha-\cos^2\beta-\cos^2\gamma-2\cos\alpha\cos\beta\cos\gamma}{\sin\alpha\sin\beta\sin^2\gamma},$$

and

$$F(c,a,b)-F(a,b,c)F(b,c,a)=\cos\gamma\,[1-F^2(a,b,c)]\,\frac{\sin\beta}{\sin\alpha}.$$

According to (4), $1-F^2(a,b,c)\neq 0$, we can solve for $\cos\gamma$:

$$(8)\qquad \cos\gamma=\frac{F(c,a,b)-F(a,b,c)F(b,a,c)}{1-F^2(a,b,c)}\cdot\frac{\sin\alpha}{\sin\beta}$$

In order to get functional equations for the unknown F-function we consider Fig. 2, in which $x > y$ and $B'D = BD = y$. DC is perpendicular to AB.

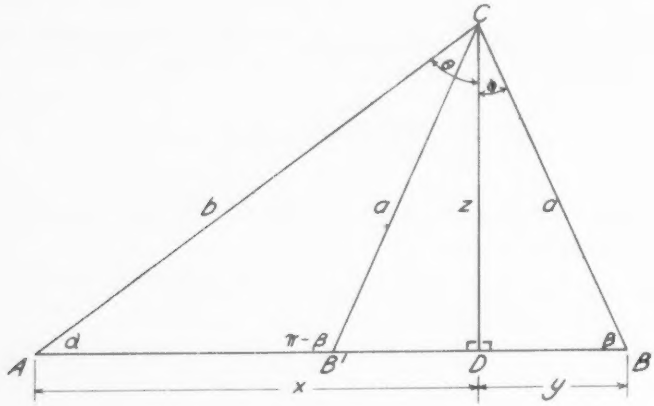

Figure 2

Using (5) we can read the following relations from the picture:

$$\frac{\cos(\theta+\phi)+\cos\alpha\cos\beta}{\sin\alpha\sin\beta} = F(x+y, a, b) \qquad \frac{\cos(\theta-\phi)-\cos\alpha\cos\beta}{\sin\alpha\sin\beta} = F(x-y, a, b)$$

$$\frac{\cos\theta}{\sin\alpha} = F(x, z, b) \qquad \frac{\cos\phi}{\sin\beta} = F(y, z, a).$$

Upon adding the first two equations together and using the identity

$$(9) \qquad \cos(\theta+\phi) + \cos(\theta-\phi) = 2\cos\theta\cos\phi$$

we find

$$(10) \qquad F(x+y, a, b) + F(x-y, a, b) = 2F(x, z, b)F(y, z, a).$$

If $x = y$, we obtain in a similar way

$$\frac{\cos 2\theta + \cos^2\alpha}{\sin^2\alpha} = F(2x, b, b) \qquad \frac{\cos\theta}{\sin\alpha} = F(x, z, b)$$

and because of the identity:

$$(11) \qquad \cos 2\theta + 1 = 2\cos^2\theta$$

we are led to:

$$(12) \qquad F(2x, b, b) + 1 = 2F^2(x, z, b).$$

Obviously the sides occurring in the functional equations (10)

and (12) are not independent of one another; and, using (6) for the two right triangles in Fig. 2, we get two more functional equations:

$$(13)\qquad \begin{aligned} F(x, z, b)\,F(z, x, b) &= F(b, x, z) \\ F(y, z, a)\,F(z, y, a) &= F(a, y, z). \end{aligned}$$

Yet the most important functional equation is obtained on substituting from (8) to (9), as follows:

$$\frac{F(x+y, a, b) - F(a, b, x+y)\,F(b, a, x+y)}{1 - F^2(a, b, x+y)} \cdot \frac{\sin\alpha}{\sin\beta}$$

$$+ \frac{F(x-y, a, b) - F(a, b, x-y)\,F(b, a, x-y)}{1 - F^2(a, b, x-y)} \cdot \frac{\sin\alpha}{\sin\beta}$$

$$= 2\,\frac{F(x, z, b) - F(z, x, b)\,F(b, x, z)}{1 - F^2(z, x, b)}\,\sin\alpha\,\frac{F(y, z, a) - F(z, y, a)\,F(a, y, z)}{1 - F^2(a, y, z)} \cdot \frac{1}{\sin\beta}.$$

In this equation we cancel at once the factor $\frac{\sin\alpha}{\sin\beta}$ and by means of (13) the right side adopts the simpler form:

$$2F(x, z, b)\,F(y, z, a)\,\frac{1 - F^2(z, y, a)}{1 - F^2(a, y, z)}$$

On using (13) again and also (10) we finally have:

$$(14)\qquad \begin{aligned} &\frac{F(x+y, a, b) - F(a, b, x+y)\,F(b, a, x+y)}{1 - F^2(a, b, x+y)} \\ &\qquad + \frac{F(x-y, a, b) - F(a, b, x-y)\,F(b, a, x-y)}{1 - F^2(a, b, x-y)} \\ &= \frac{F(x+y, a, b) + F(x-y, a, b) - 2F(a, y, z)\,F(b, x, z)\,\dfrac{F(z, y, a)}{F(z, x, b)}}{1 - F^2(a, y, z)}. \end{aligned}$$

If in like fashion we combine (8) and (11) we can get one more functional equation, but the following procedure is preferable. Assuming that $x = y$ we derive from (5)

$$F(b, x, z) = \operatorname{ctn}\theta\,\operatorname{ctn}\alpha \qquad F(b, b, 2x) = \frac{\cos\alpha + \cos\alpha\cos 2\theta}{\sin\alpha\sin 2\theta}$$

and hence

$$(15)\qquad F(b, x, z) = F(b, b, 2x).$$

The two functional equations (14) and (15) strongly suggest a special solution, so that our function F depends only on the first variable, *i.e.*

$$F(c, a, b) = f(c).$$

For this solution (14) and (15) become identities. (10) and (12) are reduced to

$$(16) \qquad f(x + y) + f(x - y) = 2f(x)f(y)$$

and

$$(17) \qquad f(2x) + 1 = 2f^2(x),$$

respectively. Since in these two equations there occur solely the independent variables x and y, it is no longer necessary to take (13) into account for their solution.

The function $f(x)$ involved in (17) and equal to $\dfrac{\cos\theta}{\sin\alpha}$ must be positive, 2θ being an angle of a triangle. We can also say, that x is one half of a segment $2x$ and in such a case $f(x) > 0$. Thus, if the segment $2x$ exists, we have

$$(18) \qquad f(x) = \sqrt{\frac{f(2x) + 1}{2}}$$

On the other hand, it follows from (17) that for *any* segment $2x$

$$(19) \qquad f(2x) + 1 > 0 .$$

To see the geometrical meaning of this inequality, we use our definition (5) and obtain

$$\frac{\cos\gamma + \cos\alpha\cos\beta}{\sin\alpha\sin\beta} > -1.$$

Hence $\cos(\alpha - \beta) > -\cos\gamma$ and

$$\alpha - \beta < \pi - \gamma \qquad \text{or} \qquad \beta - \alpha < \pi - \gamma .$$

Thus,

$$(20) \qquad \alpha + \gamma < \pi + \beta \qquad\qquad \beta + \gamma < \pi + \alpha.$$

LEMMA 2. *The sum of two angles of a triangle is always less than the third angle increased by* π.

Let us now write (7) in a simpler form

$$\frac{1 - f^2(a)}{\sin^2\alpha} = \frac{1 - f^2(b)}{\sin^2\beta} = \frac{1 - f^2(c)}{\sin^2\gamma}$$

The sides a, b, c, of a triangle can be chosen arbitrarily, whence we see that the numerators must have like signs for any a, b, c. Now,

$$1 - f^2(x) = [1 + f(x)][1 - f(x)]$$

and according to (19) $1 + f(x) > 0$. Thus $1 - f(x)$ is of the same sign for any segment x. As we have seen before, when $f(x) = 1$, $\alpha + \beta + \gamma = \pi$ and conversely. Inequality (2) shows that, if $\alpha + \beta + \gamma < \pi$, we have $f(x) > 1$. Conversely, when $f(x) > 1$, $\cos(\alpha + \beta) > -\cos \gamma$, whence

$$\alpha + \beta < \pi - \gamma \qquad \text{or} \qquad \alpha + \beta > \pi + \gamma.$$

As the latter is not possible [see (20)], there remains $\alpha + \beta + \gamma < \pi$. Finally, from $f(x) < 1$ we get $\cos(\alpha + \beta) < -\cos \gamma$ which leads to

$$\alpha + \beta > \pi - \gamma \qquad \text{or} \qquad \alpha + \beta < \pi + \gamma.$$

The second inequality is precisely (20) and we have left: $\alpha + \beta + \gamma > \pi$ [Cf. (3)]. Thus, excluding $\alpha + \beta + \gamma = \pi$, we can distinguish between:

(I) $f(x) > 1$ *HYPERBOLIC GEOMETRY* $\alpha + \beta + \gamma < \pi$

(II) $f(x) < 1$ *ELLIPTIC GEOMETRY* $\alpha + \beta + \gamma > \pi$.

Before solving the functional equations (16) and (17), it is necessary to prove that $f(x)$ is a continuous function. For this purpose we consider again the right triangle *ADC* in Fig. 2 and have:

$$\frac{\cos \theta}{\sin \alpha} = f(x) \qquad \frac{\cos \alpha}{\sin \theta} = f(z).$$

This gives rise to

$$f(x) = \frac{\cos \theta}{\sqrt{1 - f^2(z)\sin^2 \theta}}$$

Keeping z constant, the second member is, in general, a continuous function of θ. By a geometrical consideration, we see that θ is a continuous function of x and thus $f(x)$ is continuous.

We refer the reader to J. L. Coolidge, "The Elements of Non-Euclidean Geometry", Oxford, 1909, p. 52[2], where (16) together with (17) is solved. It may be useful to point out, that in (18) the square root must have a positive sign, $2x$ being any given segment. The solutions of (16) and (17) are

2. Cf. also H. E. Wolfe, "Introduction to Non-Euclidean Geometry", New York, 1945, p. 195.

(I) $f(x) = \cosh \frac{x}{K}$ for the hyperbolic case

(II) $f(x) = \cos \frac{x}{K}$ for the elliptic case.

The arbitrary constant K may be taken equal to 1 for a suitable choice of the unit of length. As it is well known, the whole trigonometry of the hyperbolic and elliptic case is now contained in (5). The functional equations admit of one more solution, *viz.* $f(x) = 1$, from which we can infer only that $\alpha + \beta + \gamma = \pi$ for the Euclidean geometry. Nevertheless this is equivalent to the Fifth Postulate and is sufficient to build up the Euclidean trigonometry. The latter may also be obtained as a limiting case for $K \to \infty$.

If in hyperbolic geometry we use for the triangle ADC the formula

$$\tan \alpha \sinh x = \tanh z$$

which can be derived easily, we are led to a formula for the parallel angle $\Pi(x)$. For as $z \to \infty$, $\tanh z \to 1$ and $\alpha \to \Pi(x)$. Thus,

$$\tan \Pi(x) \sinh x = 1.$$

For elliptic geometry, the analogous formula

$$\tan \alpha \sin x = \tan z$$

shows, that as $z \to \frac{\pi}{2}$, $\alpha \to \frac{\pi}{2}$.

No attempt has been made to find the general solution of (10) and (12), since the variables that occur are not independent, but tied together by means of (13).

University of California
Davis, California

TEACHING OF MATHEMATICS

Edited by

Joseph Seidlin and C. N. Shuster

This department is devoted to the teaching of mathematics. Thus articles on methodology, exposition, curriculum, tests and measurements, and any other topic related to teaching, are invited. Papers on any subject in which you, *as a teacher,* are interested, or questions which you would like others to discuss, should be sent to Joseph Seidlin, Alfred University, Alfred, New York.

REAL NUMBERS FOR FRESHMEN

Marie Litzinger

Mount Holyoke, like most other colleges, recently revised its curriculum. One innovation was a request that each department design a basic course for students majoring in other fields. The Department of Mathematics had for some time been discussing a revision of its offering. Finally we outlined a course which would serve both as a terminal course for majors in social studies, languages and the like, and as an introductory course for scientific students.

In our opinion a freshman course in mathematics should acquaint the student with the nature of mathematical reasoning. It should provide a careful treatment of certain concepts such as number, function, limit; and should offer an introduction to calculus. With these things in mind we planned our basic course now being taught for the second time. The first six to eight weeks are devoted to a brief discussion of the nature of a deductive system and a postulational definition of the real number system with emphasis on the rational operations and inequalities. This is followed by consideration of functions—linear, quadratic, trigonometric, and logarithmic. An introduction to differential and integral calculus completes the course.

The first year the course was given students reacted violently at the beginning. Some knew how to do these things and did not care why. Others were happy for the first time in their mathematical lives and advised their high school teachers to teach algebra this way! By the end of eight weeks almost all of them decided it was worth while. This year we were surprised by the presence of an unusual number of upper classmen in the freshman course, and by the absence of rebellion. The latter at least is due in part to the sophomores' advice "This is good stuff—you'll soon catch on." And they do. At one point they were justly annoyed at me when I asked them to prove a theorem establishing the density of the rationals. "Look," said one girl, "I can do it. But should you ask me to before you define two?" Another pointed out in my defense that they could manage by using the sum of one and one and the distributive postulate.

As to their instructors' opinion—we shall not have one of any value for several years. At the moment, we consider that the students have increased understanding of the nature of mathematics; they are more aware of the need of precision in definition and proof, and of essentials to which they return more readily when faced with new problems. Of one thing we are sure—it is fun to teach this course.

Mount Holyoke College

COMMENTS

Professor Ralph Beatley

In the colleges we have long had general mathematics in the narrow sense that the program of instruction for the freshman year—and sometimes for the sophomore year as well—has embraced two or more mathematical subjects. Those who framed the program may have intended it to be general mathematics in the wider sense also of displaying the broad basic ideas that underlie mathematics as it has developed through the ages; but whether this intent has been realized is open to question. It is possible that the better students among those who plan to concentrate in mathematics or a related field have succeeded in discovering the larger ideas in the instruction. It is probable, also, that these larger ideas have for the most part eluded the students of lesser power. All too often, gifted students with social studies or the humanities as their chief interest, but with pleasant memories of successful accomplishment in secondary mathematics and with a desire to win further insight into mathematics at the college level, have enrolled in freshman mathematics and been sadly disappointed. To them the instruction has appeared to be of interest only to the specialist. If the broad basic ideas were indeed part of the program, they received so little emphasis as to pass unnoticed by the non-specialists, and probably also by all but the most gifted specialists. Add to these the less gifted non-specialists for whom faculty committees and deans desire some broadening experience of college mathematics, but "without tears", and we have the reason why the colleges are now devising new courses in general mathematics at the college level.

The intent of the new courses is to minimize technical details and to emphasize the chief characteristics of different branches of mathematics so far as they are of interest to the general student and lie within his powers of understanding. Three questions immediately arise: first, just what are the larger ideas of mathematics; second, what is the best way to present these to freshmen and sophomores; and third, will experience with general mathematics designed for the non-specialist

suggest materials or methods that could be adapted for use in the instruction of specialists?

Both Professor Litzinger and Professor Brown would include some of the infinitesimal calculus in a general course, and many college teachers would agree with them. With respect to other topics there is difference of opinion, and difference of opinion also with respect to the details of the calculus.

Professor Litzinger would begin the course with a concise and precise postulational development of the real number system. She regards this as necessary to a proper treatment of the subsequent section on the calculus. Presumably also she values the real number system for its own sake, as one of the basic ideas that a course of this sort ought to include. Many would agree with her on this and yet not put it first, nor require it for the section on the calculus. It may be pertinent to comment that even for specialists the calculus is rarely begun that way today in this country.

It seems proper to consider the interests of the pupil and how best to motivate him to accept the ideas we wish to lay before him. Even if it is possible to begin "the hard way", with one of the least fascinating of all the topics in the program, is it not good strategy to try to make the first steps attractive—and later steps also, so far as possible? By so doing we might win more easily the student's active participation in the course, and so increase the profit he derives from the instruction.

In addition to the calculus and the real number system Professor Brown suggests topics from analytic geometry; probability and statistics; cartography; and the algebra of logic. Some topics can serve two or more purposes at once. The real number system is one example of this. The Boole-Schroeder algebra of classes is another. The latter exemplifies a postulational system; also it exhibits a valid, but "unorthodox" algebra to parallel non-Euclidean geometry; and it can be interpreted as an algebra of logic, the logic that we use in all our mathematical thinking.

Every department of mathematics will have its own preferences, and its own way of ordering them. Probably the topics most frequently preferred will be those mentioned by Professor Litzinger and Professor Brown. Professor A. A. Bennett expressed the opinion that the particular topics selected are relatively unimportant; that the important thing is to secure instructors who are at home in their subject, sympathetic with the purposes of the general mathematics program, and interested in the pupils who face them in class. Such instructors will speedily discover what topics are suitable and what methods work best. Such a point of view has much to commend it.

Harvard Graduate School of Education

(See also Professor Seidlin's "comments", page 156, Mathematics Magazine, for Jan.-Feb., 1949. *Ed.*)

SOME CONCEPTS OF ELEMENTARY TOPOLOGY

Dick Wick Hall

Everyone knows what a circle is. It consists of all points in a plane at a fixed distance from a given point, called the center. Figure 1 represents a circle with center at the point O and radius unity. This circle will be quite important to us, and we wish to be on very friendly terms with it. We shall call it the circle K, to give it a convenient name. Take a good look at the points Y and Z on K. Notice that there are two arcs of K joining Y and Z. One is the arc YXZ, the other is the arc YWZ. These arcs have no points in common except their end points. Every point of K is on either the arc YXZ or the arc YWZ.

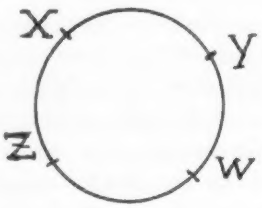

Figure 1

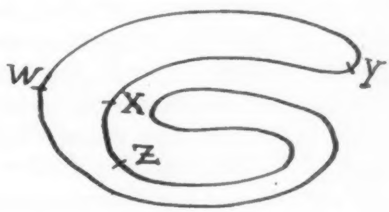

Figure 2

In plane geometry you studied circles and other geometric figures and learned a great many useful facts about them. Topology continues the study of geometry, but goes at it from an entirely different point of view. In the few pages at our disposal here we shall try to give a glimpse, if but a fleeting one, of the topologist's viewpoint. Further information can be obtained from any of the standard elementary textbooks on the subject.

The topologist might well begin studying circles by finding himself a rubber band. Take a small one from your desk and lay it down on top of the circle K. Imagine that luck has been with you, and that the rubber band fits the circle K exactly. The topologist calls the rubber band a simple closed curve because it can be made to coincide with the circle K

Call the rubber band J. When last seen J was laying down on top of K.

Pick it up and place it somewhere else in the plane. Don't worry about bending it, or stretching it, but be careful that it does not break. Also, do not allow any two of its points to touch each other. The rubber band J, in its new resting place, may look something like Figure 2. We have kept the letters the same so that it will be a simple matter to decide where the points of the curve J came from. The point labeled W, for example, is the point which was on top of the point W on the circle K before we moved J.

No decent geometry teacher would call J a circle. It isn't even "round". It doesn't have a center or a radius. It does have exactly two arcs joining its points Y and Z; these arcs have no points in common except their end points; while every point of J is on one of these two arcs. These arcs, however, if they are of finite length, certainly will not have the same lengths as the original arcs of K. Or they might! Depending upon how the bending was done it can easily be shown that almost any statement which one might make about the lengths of the original arcs would be false for at least one type of bending. The topologist, however, still calls J a simple closed curve.

Let us formalize a trifle what we have done in the above bending process. We do this by saying that the circle K has been mapped onto the simple closed curve J. The mapping is indicated by writing $f(K) = J$. This means that to each point x of K there corresponds exactly one point $f(x)$ of J, namely the point of J which was originally on top of the point x of K. The mapping $f(K) = J$ is said to be single-valued, since to each point x of K there exists exactly one point $f(x)$ of J. It is said to be one-to-one since no two points of K map into the same point of J. The mapping f also has a property corresponding to our agreement not to allow the rubber band to break. This property is called continuity. It will be mentioned again near the end of this paper. *A transformation $f(K) = J$ is a homeomorphism* means that it is single-valued, one-to-one, and continuous in both directions. The "in both directions" means that if we move the rubber band back onto K from its position in Figure 2, the backward movement will not break the rubber band.

Topologists call simple closed curves any sets J which can be obtained from K by means of a homeomorphism $f(K) = J$. They are interested primarily in properties of sets which do not change under homeomorphisms, and it is for this reason that such properties are commonly referred to as "topological properties".

Let us study a few topological properties of simple closed curves. Before taking up the "nice" properties of simple closed curves, it might be well to point out how bad a simple closed curve may become. Those of you interested should read a very interesting and elementary paper by J. R. Kline. (See American Mathematical Monthly, Vol. 49 (1942) pp. 281-285). Also highly recommended is "Mathematics and the Imagination" by Kasner and Newman. Professor Kline shows, in his paper, how to construct simple closed curves of infinite length which have no

tangent at any point. He also shows how a simple closed curve may be constructed which has a two-dimensional area (measure) greater than zero. Thus the so-called "simple" closed curves can become quite rugged animals, and it is indeed remarkable that they retain the property of a circle which we next consider.

One of the very important properties of a circle is that it cuts the plane into exactly two parts. The one of these that contains the center is called the "inside", the other one is called the "outside". We say that a circle has exactly one inside and exactly one outside. One interpretation of this statement consists of the following three assertions: (A) If P and Q are any two points inside a circle, then there is an arc from P to Q in the plane such that every point of the arc is inside the circle. (B) If P and Q are any two points outside a circle, then there is an arc from P to Q in the plane such that every point of the arc is outside the circle. (C) If P is any point inside a circle and Q is any point outside the circle then every arc from P to Q in the plane must intersect the circle.

Very few people worried about simple closed curves until the nineteenth century when the French mathematician Jordan realized that there was definitely a problem (a hard one too, if one takes the usual axioms of plane geometry) in proving that every simple closed curve J satisfies statements (A), (B), (C) of the previous paragraph in which the word "circle" has been replaced by "simple closed curve". His theorem is as important and fundamental to the topologist as is the Pythagorean theorem to those studying geometry by means of distances. The Jordan curve theorem tells us not only that (A), (B), and (C) hold, but also that (D) If P is any point on a simple closed curve J then there is a point A outside of J and a point B inside of J such that the line segment from A to P meets J only in P while the line segment from B to P meets J only in P.

This may all sound quite "obvious" and foolish to you, but quite often we overlook the "obvious" things. Perhaps in your high school days you fooled around with the problem of the three farmers A, B, C and the three wells H, K, L. It seems that each of the farmers wished to run a pipe to each of the three wells. This would have been easy had it not been for the additional restriction that all the pipes were to lie in the same plane and no two of them were to cross. You undoubtedly got as far as having pipes from A and B to each of the three wells H, K, L. Let us see what your diagram looked like. In Figure 3 you see that the pipes AH, AL, BH, BL, form a simple closed curve and that K is inside that simple closed curve. If C is outside this simple closed curve, then C cannot run his pipe to K. But if C is not outside this simple closed curve, then C must

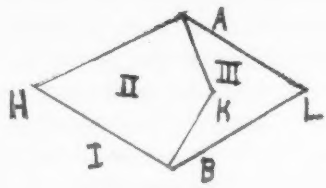

Figure 3

be either in the region II or the region III. If C is in II, then no pipe can be run from C to L, while if C is in III, then no pipe can be run from C to H. As an exercise, find the simple closed curve which does the separating in each case.

There are closed curves in the plane which divide it into exactly two parts, but which are not simple closed curves. One of them is the example which has been used by every decent topologist at least once. (Topologically, in fact, this is almost a definition of decency!) You can easily draw the curve yourself with what you remember of analytical geometry. It consists of the set composed of the following four parts (1) $y = \sin(\pi/x)$, $(0 < x \leq 1)$, (2) the interval of the y-axis from $(0, -2)$ to $(0, 1)$, (3) the interval of $y = -2$ from $(0, -2)$ to $(1, -2)$, (4) the interval of $x = 1$ from $(1, -2)$ to $(1, 0)$. You will notice that this curve wiggles between the lines $y = +1$ and $y = -1$. Draw enough of it to convince yourself that the closer you get to the origin the faster it wiggles. As a matter of fact, it wiggles from $y = +1$ to $y = -1$ infinitely many times on its way to the y-axis (which it never reaches).

You can easily see that this "curve" cuts the plane into exactly one inside and exactly one outside. Now go back to the Jordan curve theorem, and look at conditions (A)—(D). You will have little difficulty convincing yourself that, for the curve just drawn (A), (B), and (C) are true. However, condition (D) is false. If B is any point inside the curve it is impossible to draw an arc from the point B to the point O (the origin) which meets the curve only at O. It looks like this can be done, but try it! Starting at B and going towards O one must eventually go down into one of the infinitely many dips of the curve. The left side of the dip then prevents you from going straight in to O, hence you must come out of the dip. But if you must come out, why go in? It is a hopeless situation, and O cannot be reached. We say that O is not accessible from the inside of the curve.

The reason that O is not accessible from the inside of the closed curve may be expressed roughly by saying that the curve wiggles too much. To put this into more exact terms, notice that we can find two points outside the curve, call them A and B, which are near the bottoms of two different dips and yet are as close together as we want them to be. We could choose them closer than a thousandth of an inch, for example. To join these points by an arc outside the curve would necessitate going all the way out of the first dip and all the way back into the second one. Thus the arc joining A and B must be almost four inches long (we are using the inch as our unit of measurement on the x and y-axis), even though the points A and B were extremely close together to begin with. Here lies the trouble! Here indeed is the roach in the raspberries. It can be proven that if e is any positive number whatever (think of one), then a number d can be found (depending very strongly on e) such that if A and B are any two points both inside the given simple closed curve J and whose distance apart is less than d, then

there exists a simple arc from A to B inside J and of length less than e. This same result holds, of course, for the outside of J. We thus see that, although a simple closed curve may wiggle badly enough to cover a set of points of positive two-dimensional measure, it still cannot wiggle badly enough to prevent accessibility. There is no wonder that these strange creatures are so universally admired by topologists.

We have seen above that a simple closed curve J has exactly one inside and exactly one outside. Each of these two sets may be regarded as being "connected" if we regard (A) and (B) as the definitions of connectivity. It is one of the chief delights of the mathematician, however, to generalize. Let us look at another example and then try to get a more general definition of a connected set than the one given in (A).

To picture our example recall that the curve that wiggled too badly was composed of four parts. The example will be a set we call F, composed of the first two of these parts. Thus F consists of a part A which is an interval of the y-axis, and a part B which is part of our particular sine curve. Do we want to call F a connected set? It is evident that we cannot join a point of A to a point of B by an arc entirely in F. Thus F is certainly not "arcwise connected". However, there is no possibility of dividing F into two pieces in such a way that we could go between them on perhaps a simple arc. There seems to be no simple closed curve in the plane which has the property that it does not intersect F, but has A on its inside and B on its outside. For this reason F appears to be connected. Let us try to define connectivity so that F will be connected under our definition.

Notice that if we divide F into the two pieces A and B that the point O is in A. And yet, for any positive number e, there exists a point of B whose distance from O is less than e. For this reason we say that O is a limit point of B.

The idea of limit point is so fundamental in topology that it is perhaps worth while to spend a moment or two more on its definition. The point P is said to be a limit point of the set A provided that for any positive number e there exists a point Q of A, different from P, such that the distance from P to Q is less than e. If one considers the set A as consisting of the points on the x-axis having abscissae $1, 1/2, 1/3, \cdots 1/n, \cdots$ then the origin is a limit point of the set A. If, on the other hand, A consists of the entire x-axis, then every point of A is a limit point of A. A set which contains all of its limit points is said to be closed.

Before returning to the idea of connectedness we mention a very important theorem on simple closed curves. It can be shown that any closed and bounded subset of the Euclidean plane satisfying conditions (A)—(D) of the Jordan curve theorem is a simple closed curve. By this is meant that for any such set J there exists a mapping $f(K) = C$ which is a homeomorphism. Here K denotes the unit circle, defined at the

beginning of this paper.

To return to connectedness, we shall call a set A connected provided that no matter how we divide it into two subsets B and C such that if (i) B is not empty, (ii) C is not empty, (iii) every point of A is either a point of B or a point of C, (iv) every point of B and every point of C is a point of A, (v) B and C have no points in common, then either B contains a limit point of C or C contains a limit point of B.

There are many ways of using the idea of a connected set in mathematics. One of the places where it is most convenient is in setting up the real number system. At the point where the rational numbers have been defined some device or other must be introduced in order to obtain the irrationals. One rather "natural" axiom to take is that the x-axis (the axis of real numbers) is a connected set. It is then a simple matter to prove the Dedekind Cut Postulate.

Dedekind Cut Postulate: *If the set of all points on the x-axis is divided into two non empty subsets L_1 and L_2 which have no points in common and such that for any pair of points x in L_1, y in L_2 we have x is less than y, then there exists a point d which is either the last point of L_1 or the first point of L_2.*

Proof: Since the line is a connected set, either L_1 must contain a limit point of L_2 or L_2 must contain a limit point of L_1. The proofs in the two cases being similar, we may suppose L_1 contains a point d which is a limit point of L_2. If d is not the last point of L_1, then, there exists a point d' of L_1 greater than d'. Thus no point of L_2 has a distance from d less than $d' - d$. This contradicts the fact that d is a limit point of L_2, and completes the proof of the theorem.

A moment's reflection shows that if A is a set of real numbers all less than some number r, then no number s greater than r is a limit of A. (For no number of A is closer to s than $s - r$). In exactly the same way if all numbers of A are greater than r, then no number less than r is a limit point of A. We see easily from these two statements that if A is a connected subset of the real axis and if P and Q are any two points of A then A contains every point between P and Q.

There are very few sets on the x-axis which have the property just mentioned. Every connected set has this property, and we leave it to the reader to supply the easy proof of the fact that every set having this property is connected. Knowing this we may list the different types of non empty subsets of the line as follows:

(*a*) the entire line.
(*b*) the closed interval joining two points.
(*c*) the half-open interval joining two points, *i.e.* the closed interval with either end point omitted.
(*d*) the open interval joining two points, *i.e.* the closed interval with both end points omitted.
(*e*) the closed ray, consisting of a point and all points to the

right of this point; or the point and all points to the left of this point.

(f) the open ray, consisting of a closed ray with the end point omitted.

At the very beginning of this paper we said a few words about a mapping $f(K) = J$ where K was the unit circle and J was any simple closed curve. We would like to close the paper by saying a few more things about mappings. This time we consider two sets A and B and a mapping $f(A) = B$. This mapping will have the following properties: (i) it is single-valued, *i.e.* to each point a of A there corresponds one and only one point b of B. The point corresponding to a given point a of A is denoted by $b = f(a)$. (ii) to every point b of B there corresponds at least one (possibly very many) points a of A such that $b = f(a)$. (iii) $f(A) = B$ is continuous. To explain precisely what this means we recall that a set is closed if and only if it contains all of its limit points. A subset K of B is said to be closed in B provided that K contains all of its limit points which are in B. Given a subset K of B we denote by $f^{-1}(K)$ the subset of A consisting of all points of A which map into points of K. The statement $f(A) = B$ is continuous means precisely that for any subset K of B which is closed in B, $f^{-1}(K)$ is closed in A.

Suppose now that $f(A) = B$ satisfies the three conditions of the previous paragraph and that A is connected. We shall prove that B is connected. If this were not the case we could divide the points of B into two non empty subsets C and D satisfying the conditions (i) C and D have no points in common, but between them contain all points of B and no other points, (ii) C contains no limit point of D, and D contains no limit point of C.

Condition (ii) says that each of the sets C and D is closed in B. Thus each of the sets $f^{-1}(C)$ and $f^{-1}(D)$ is closed in A. Thus the points of A can be divided into the two sets $f^{-1}(C)$ and $f^{-1}(D)$, neither of which is empty and each of which is closed in A. This tells us at once that A is not connected, which is a contradiction to our original assumption. The proof that B is connected is thus complete.

Quite commonly in mathematics we deal with continuous functions $y = f(x)$ of a real variable. Suppose we have such a function defined on a connected subset A of the x-axis. This means that to each real number x in the set A there corresponds a real number y such that $y = f(x)$. We may consider the set of all real numbers y which are images of points of A under f to be a set of points B on the y-axis. Thus our mapping $y = f(x)$ becomes a continuous mapping $f(A) = B$. Since A is connected, B must be connected. But this means that if P and Q are any two points of B then B contains every point between P and Q. Stated in more familiar language, if x_1 and x_2 are points of A and if $y_1 = f(x_1)$, $y_2 = f(x_2)$ then for any real number y_3 between

y_1 and y_2 there exists a real number x_3 in A such that $y_3 = f(x_3)$. This is the familiar intermediate value theorem for continuous function. It is used by every freshman in his algebra course when he tries to find approximate roots of equations. Suppose he is required to approxmate the roots of $f(x) = 0$. If he knows that $f(1)$ is positive, while $f(2)$ is negative, then, by the above theorem, there must be a root of $f(x) = 0$ between 1 and 2.

The brief sketch above is meant to give a small indication of the type of material enjoyed by the average topologist. If this kind of mathematics interests you, do not fail to investigate it further.

University of Maryland

MATHEMATICAL MISCELLANY

Edited by

Marion E. Stark

Let us know (briefly) of unusual and successful programs put on by your Mathematics Club, of new uses of mathematics, of famous problems solved, and so on. Brief letters concerning the MATHEMATICS MAGAZINE or concerning other "matters mathematical" will be welcome. Address: MARION E. STARK. Wellesley College, Wellesley, 81, Mass.

The letter of the month:

An historical investigation in which we have been engaged for some years impressed upon us the fact that mathematics came into being and has developed as a universal language.

Certainly a project that seeks to conserve the intelligibility of mathematics in a time of proliferating specialization deserves support.

I will be very much interested to see how you go about the job, which must have seemed fantastic to mathematicians of even a century ago, of making mathematics intelligible to mathematicians! There is nothing fantastic about making mathematics intelligible to non-mathematicians except the difficulties encountered by the innocent teacher; but I do think that mathematicians of Newton's time, or even later, would find elements of fantasy in a situation where competent mathematicians must confess that they do not understand the mathematical languages spoken by other mathematicians.

But where is the distinction or line of demarcation between a "mathematician" and a "non-mathematician"? Is the criterion a knowledge of mathematics or the love of mathematics? If the former, how much knowledge makes a "mathematician"? Is it necessary to have mastered the integral and differential calculus before being admitted to the circle of initiates? If so, how do we deal with Archimedes and Euclid who, without benefit of calculus, were mathematicians of the first rank? Where, I ask your readers, does the "non-mathematician" end, and the "mathematician" begin? In mathematics we proceed from definitions. Will someone please define a mathematician.

Just to muddy the waters I will submit two contributions. A dear friend of mine who died last year was the most eminent scientist in the world in his very special field of study, and much of his work was mathematical, yet he confessed to me that he found mathematical reasoning an agony and abhorred the subject. Would it be correct to call him a "mathematician"?

The world's first school of mathematics was founded nearly twenty-five centuries ago at Croton, Italy by Pythagoras of Samos. The initiates of this school were the first men to call themselves "Mathematicians".

It is said that a man could not earn this degree until he proved by test that he possessed a mature and self-critical sense of humor. This idea of defining a "Mathematician" by sense of humor is not without merit, although if the same test were applied to modern candidates for a doctor's degree in mathematics it might seriously cut down the supply of accredited mathematicians.

After reading three issues of *Mathematics Magazine* I wrote to Professor Glenn James and expressed my appreciation of the job you are all doing by sending my check for a sponsor's subscription. So long as you hold to the Pythagorean rules that mathematics should be served with love, and that mathematicians should not take themselves or their science *too* seriously, you will have my hearty support.

American Institute of Man Alexander Ebin

Editorial Comment.

We hope, Mr. Ebin, that some of our readers will send us discussions of the questions you have raised. But your letter also makes almost necessary an immediate statement concerning the policy of the Mathematics Magazine in the matter in which you show such kindly interest.

The symbolic language of mathematics does, as you suggest, isolate most of mathematics from the general public and some highly specialized fields from some highly specialized mathematicians. This is a situation fraught with danger to mathematics (as evidenced by the de-emphasizing of mathematics in the secondary schools, and exceedingly inconvenient to the "intelligent public" who are interested in the principles of mathematics and or need it in their daily living.

One of the major objectives of the Mathematics Magazine is to do some effective work in bridging this gulf.

Our first step has been to begin the publication of a series of chapters on "The Meaning of Courses in Mathematics" or as some prefer to call them "Understandable Chapters in Mathematics".

The next step which we are just undertaking with this issue is to encourage our authors to precede their usual introductions with *forewords* telling in simple language what their papers are all about.

Perhaps these forewords will be too technical; perhaps they will be too simple. At any rate they are difficult to write, and your opinion and the opinions of other readers about them, as they appear from time to time will be much appreciated.

G. J.

Two Curious Typographical Errors

On page 130 of SCIENTIFIC ABBREVIATIONS, SIGNS AND SYMBOLS, Industrial Research Service, Dover, New Hampshire, 1948, occurs the

following definition:

—Viniculum · · · ·	indicate that the quantities to which they are applied,
() Parentheses : · ·	or which are enclosed by them, are to be taken together,
· · · · · · · · · · ·	*i.e.*—treated as a single number.

Now vinic pertains to wine or alcohol and viniculum might be considered a container of such beverages in which case the definition is not altogether inappropriate!

On page 28 of a CONCISE HISTORY OF MATHEMATICS - Dover, 1948, will be found the following sentence:

"The texts show that the Babylonian geometry of the Semitic period was in possession of formulas for the areas of simple rectilinear figures and for the volumes of simple solids, though the volume of a *frustrated* pyramid has not yet been found"!

Of course a truncated pyramid is certainly frustrated!

P. D. Thomas

A Simple Method for Approximating Logarithms

There may be some value in including in the college algebra course an approximate method for calculating logarithms based on concepts developed in that course. A simple table of logarithms, generally good to two places, can be constructed without using ideas beyond the definition of logarithms itself; with the additional aid of interpolation it is possible to obtain values correct to four or more places without difficulty.

To build a two-place table, we note that 2^{10}, i.e., 1024, is approximately equal to 10^3; thus, approximately:

$$10 \log 2 = 3 \log 10, \quad \log 2 = .30 .$$

Similarly, 81 is nearly 80, 49 nearly 50, 121 nearly 120, and so on, giving:

$$4 \log 3 = 3 \log 2 + \log 10, \quad \log 3 = .48,$$
$$2 \log 7 = 2 \log 10 - \log 2, \quad \log 7 = .85,$$
$$2 \log 11 = 2 \log 2 + \log 3 + \log 10, \quad \log 11 = 1.04, \text{ etc.}$$

For more accurate values, we can write 1024 as 1000×1.024, 81 as 80×1.0125, and so on, find approximate values for log 1.024, log 1.0125, and the like, and use these values in calculating log 2, log 3, etc. Instead of trying to approximate the logarithms of 1.024, 1.0125, etc., directly, we can find logarithms of numbers close to these numbers and then use interpolation to get the desired results.

This may be done in several ways. Perhaps the simplest is to calculate log 1.25, using the value of log 2 already calculated, and interpolate between unity and 1.25. This procedure gives $\log 1.024 = .01$, $\log 1.0125 = .005$, $\log .98 = -.008$, and, using these values, $\log 2 = .301$, $\log 3 = .477$, $\log 7 = .845$, all correct to three places. More accurate results can be obtained by observing that $(1.01)^{12}$, $(1.02)^6$, and $(1.03)^4$ are all nearly equal to 9/8. Using the values already obtained to

calculate log 9/8, we get log 1.01 as .0043, log 1.02 as .0085, log 1.03 as .0128. Basing our interpolation on these values, we get values for log 2, log 3, etc. good to four places. Repeating this process, using the values last calculated to recompute the logarithms of 1.01, 1.02, and 1.03, we obtain approximations for the logarithms of 2, 3, 7, etc. correct to five places.

A different method which avoids the question of the convergence of the successive approximations involved in the method of the last paragraph consists in basing the interpolation on the 64th, 128th, and 256th roots of ten and the logarithms of these numbers. The actual computation of these roots is not nearly as laborious as one might assume. Since we may obtain these numbers as accurately as is desired and since their logarithms are known exactly, the only factor contributing to the inaccuracy of this method is the use of linear interpolation. The effects of this factor may be reduced by decreasing the range of interpolation by the use of suitable combinations of the roots of 10. Thus, if we interpolate for log 1.024 between the logarithms of $10^{1/128} \times 10^{1/512}$ and $10^{1/128} \times 10^{1/512} \times 10^{1/1024}$, we obtain log 2 correct to eight decimal places.

Rutgers University Edmund Churchill

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